Examples of fitting various piecewise-continuous functions to data, using basis functions in doing the regressions.

Change log:

04Oct24: Added the case of a bilinear function in which the slope of the second segment is constrained (this arose in regressions of NGA-W3 data for geometrical spreading).

David M. Boore

These examples in this document used R to do the regression. See also *Notes_on_piecewise_continuous_regression.doc* for more detail on why the basis functions used below guarantee continuity at the breakpoints.

```
###################################
# Based on Eric Thompson's program piecewise-continuous.R, using DMB basis
functions

#-----------------------------------------------------------------------------
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#-----------------------------------------------------------------------------
-----------
Title <- "QUADRATIC, LINEAR, LINEAR"
# Define function
# quadratic, M<4, linear, M 4 to 7, linear M>7

# Breakpoints:
c <- c(4, 7)
y <- function(M,c){
  ifelse(M<c[1],1.13+1.4*(M-c[1])-0.5*(M-c[1])^2,
    ifelse(M<=c[2],1.13+0.5*(M-c[1]), 1.13+0.5*(c[2]-c[1])-0.5*(M-c[2])))
 }

#Generate data
set.seed(1)
n <- 300
M <- runif(n, 0, 10)
Msorted <- sort(M)

# Add some noise:
yn <- y(M,c) + rnorm(n, sd = 0.5)

plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)
```
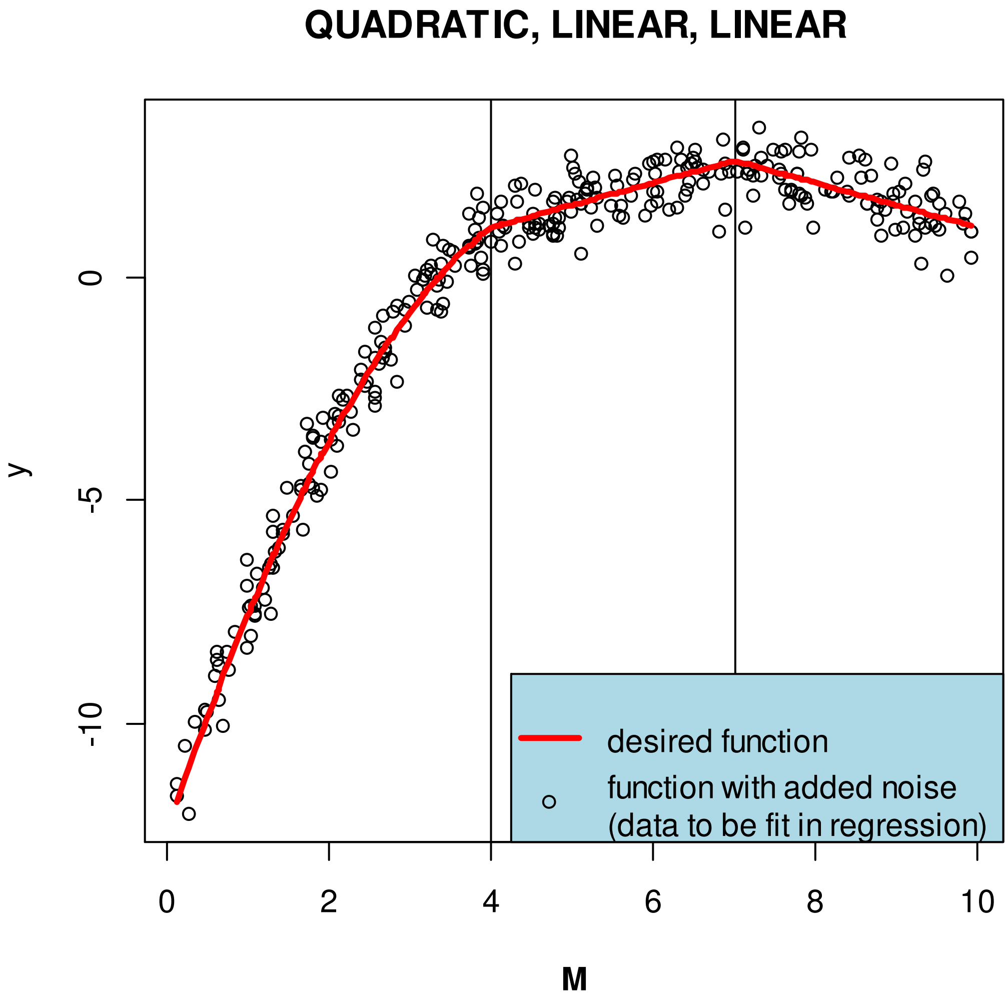
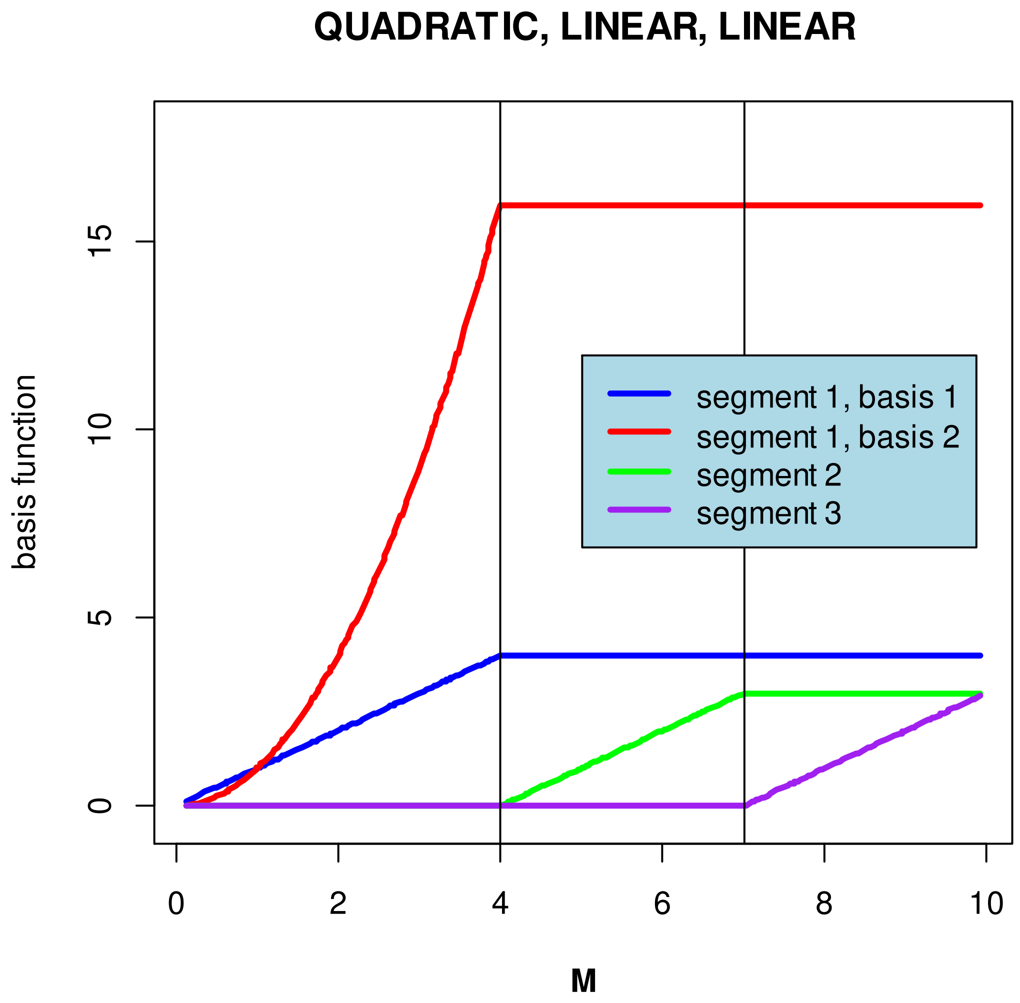

```
abline(v = c)
lines(Msorted, y(Msorted,c),lwd=3, col="red")
legend("bottomright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)"),
        lty=c(1,NA), lwd=c(3,NA), pch=c(NA,1), col=c("red", "black"),
bg="lightblue")
```



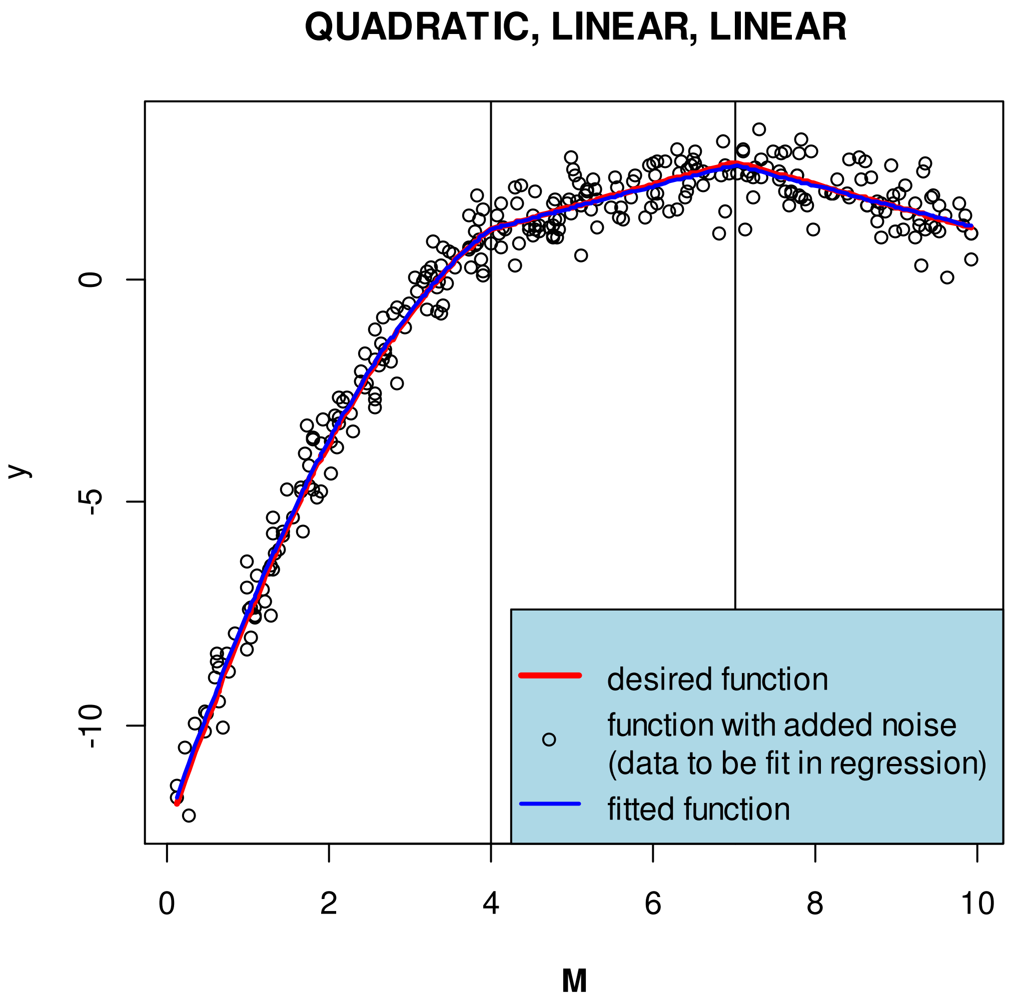
```
plot(Msorted, b1(Msorted,c[1]), xlab=substitute(paste(bold("M"))),
ylab="basis function",
  ylim=c(-0.25,18.0), type="l", col = "blue", lwd=3, main=Title)
lines(Msorted, b1.2(Msorted,c[1]), col = "red", lwd=3)
lines(Msorted, b2(Msorted,c[1],c[2]), col = "green", lwd=3)
lines(Msorted, b3(Msorted,c[2]), col = "purple", lwd=3)
abline(v = c)
```

```
legend(5,12, legend=c("segment 1, basis 1","segment 1, basis 2", "segment 2",
"segment 3"),
        lty=rep(1,4), lwd=rep(3,4), col=c("blue", "red", "green", "purple"),
bg="lightblue")


# Compute basis function for each region:
b1 <- function(x,R){ifelse(x<R,x,R)}
b1.2 <- function(x,R){ifelse(x<R,x^2,R^2)}
b2 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,x-L,R-L))}
b3 <- function(x,L){ifelse(x<L,0,x-L)}
# NOTE: L and R are not the necessarily the same for all segments; they are
the breakpoints on either side of each segment.
# See the call to lm to see what values of L and R are used for each basis
function.

plot(Msorted, b1(Msorted,c[1]), xlab=substitute(paste(bold("M"))),
ylab="basis function",
  ylim=c(-0.25,18.0), type="l", col = "blue", lwd=3, main=Title)
lines(Msorted, b1.2(Msorted,c[1]), col = "red", lwd=3)
lines(Msorted, b2(Msorted,c[1],c[2]), col = "green", lwd=3)
lines(Msorted, b3(Msorted,c[2]), col = "purple", lwd=3)
abline(v = c)
legend(5,12, legend=c("segment 1, basis 1","segment 1, basis 2", "segment 2",
"segment 3"),
        lty=rep(1,4), lwd=rep(3,4), col=c("blue", "red", "green", "purple"),
bg="lightblue")
```

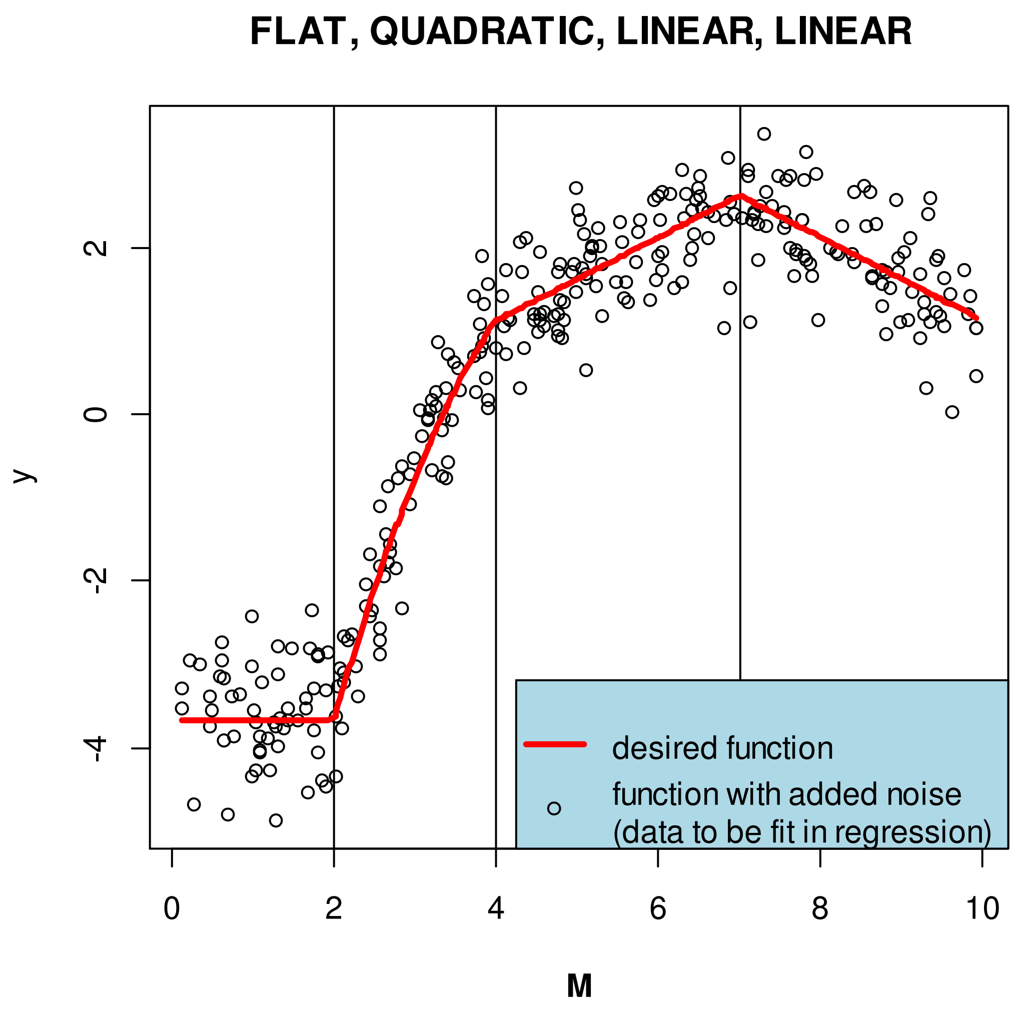


```
(NOTE: new figure made on 2022-10-18 using ylim chosen to show the b1.2
basis)

Model <- lm(yn ~ b1(M,c[1]) + b1.2(M,c[1]) + b2(M,c[1],c[2]) +  b3(M,c[2]) )
# Regression summary:
summary(Model)
#Call:
#lm(formula = yn ~ b1(M, c[1]) + b1.2(M, c[1]) + b2(M, c[1], c[2]) +
#    b3(M, c[2]))
#
#Residuals:
#     Min       1Q   Median       3Q      Max
#-1.44527 -0.30024 -0.01794  0.34079  1.25788
#
#Coefficients:
```

```
#                    Estimate Std. Error t value Pr(>|t|)
#(Intercept)       -12.31402    0.17147 -71.814   <2e-16 ***
#b1(M, c[1])         5.33851    0.16617  32.127   <2e-16 ***
#b1.2(M, c[1])      -0.49422    0.03438 -14.374   <2e-16 ***
#b2(M, c[1], c[2])   0.47826    0.04176  11.454   <2e-16 ***
#b3(M, c[2])        -0.46825    0.05141  -9.109   <2e-16 ***
#---
#Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#
#Residual standard error: 0.5128 on 295 degrees of freedom
#Multiple R-squared:  0.9795,    Adjusted R-squared:  0.9793
#F-statistic:  3529 on 4 and 295 DF,  p-value: < 2.2e-16#Coefficients:

coef <- coefficients(Model)
#> coef
#      (Intercept)       b1(M, c[1])     b1.2(M, c[1]) b2(M, c[1], c[2])
b3(M, c[2])
#      -12.3140225         5.3385115        -0.4942249         0.4782630
-0.4682550

# Replot data
plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)
# Plot actual function
abline(v = c)
lines(Msorted,y(Msorted,c), lwd=3, col="red")

# Add predictions
yp <- coef[1]+coef[2]*b1(Msorted,c[1])+coef[3]*b1.2(Msorted,c[1]) +
      coef[4]*b2(Msorted,c[1],c[2]) +
      coef[5]*b3(Msorted,c[2])
lines(Msorted,yp, lwd=2, col="blue")
legend("bottomright", legend=c("desired function", "function with added
  noise\n(data to be fit in regression)", "fitted function"),
  lty=c(1,NA,1), lwd=c(3,NA,2), pch=c(NA,1,NA), col=c("red", "black",
  "blue"), bg="lightblue")
```

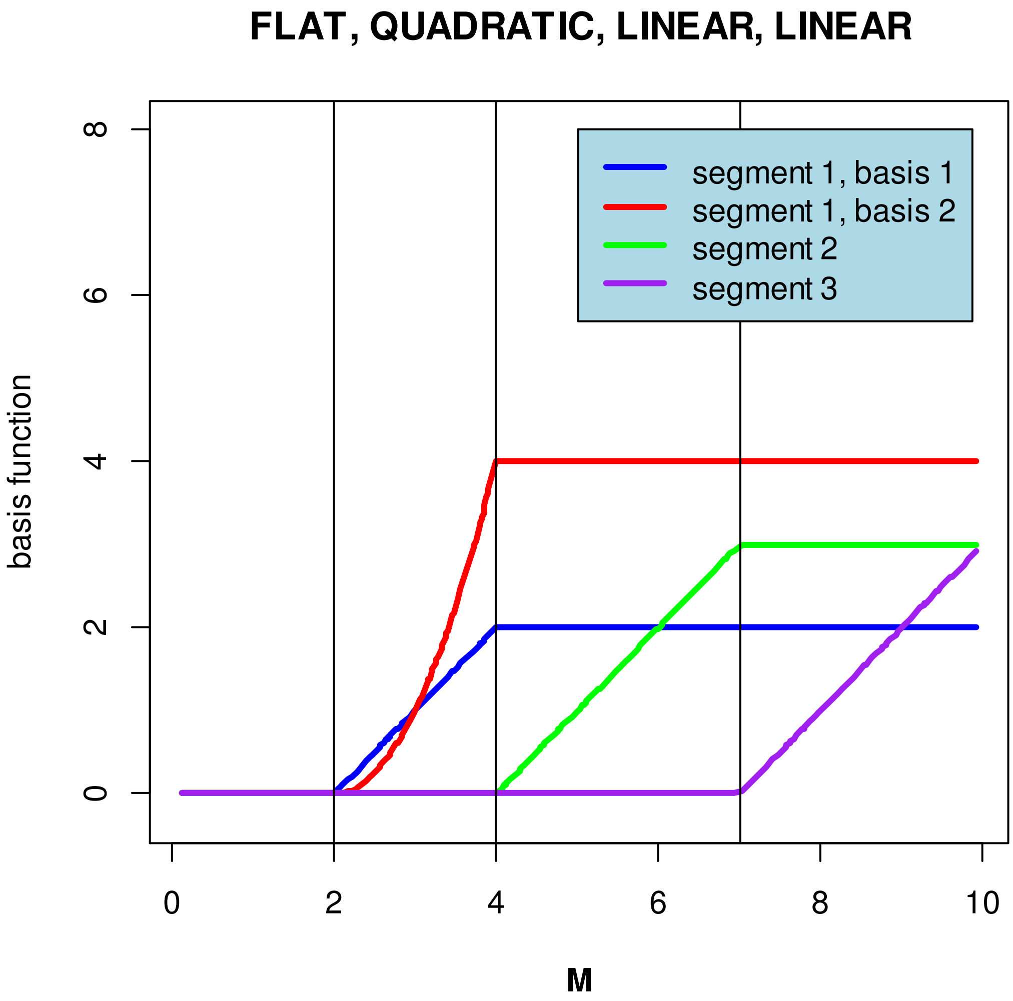

QUADRATIC, LINEAR, LINEAR
y
0
-5
-10
0
2
4
6
8
10
M
desired function
function with added noise
(data to be fit in regression)
fitted function

```
Title <- "FLAT, QUADRATIC, LINEAR, LINEAR"
# A basis function is not needed for a flat segment
# Define function

# Breakpoints:
c <- c(2, 4, 7)
y <- function(M,c){
  ifelse(M<c[1], -3.67,
    ifelse(M<c[2],1.13+1.4*(M-c[2])-0.5*(M-c[2])^2,
      ifelse(M<=c[3],1.13+0.5*(M-c[2]), 1.13+0.5*(c[3]-c[2])-0.5*(M-c[3]))))
 }


#Generate data
set.seed(1)
n <- 300
M <- runif(n, 0, 10)
Msorted <- sort(M)

# Add some noise:
yn <- y(M,c) + rnorm(n, sd = 0.5)

plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)
abline(v = c)
lines(Msorted, y(Msorted,c),lwd=3, col="red")
legend("bottomright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)"),
        lty=c(1,NA), lwd=c(3,NA), pch=c(NA,1), col=c("red", "black"),
bg="lightblue")
```



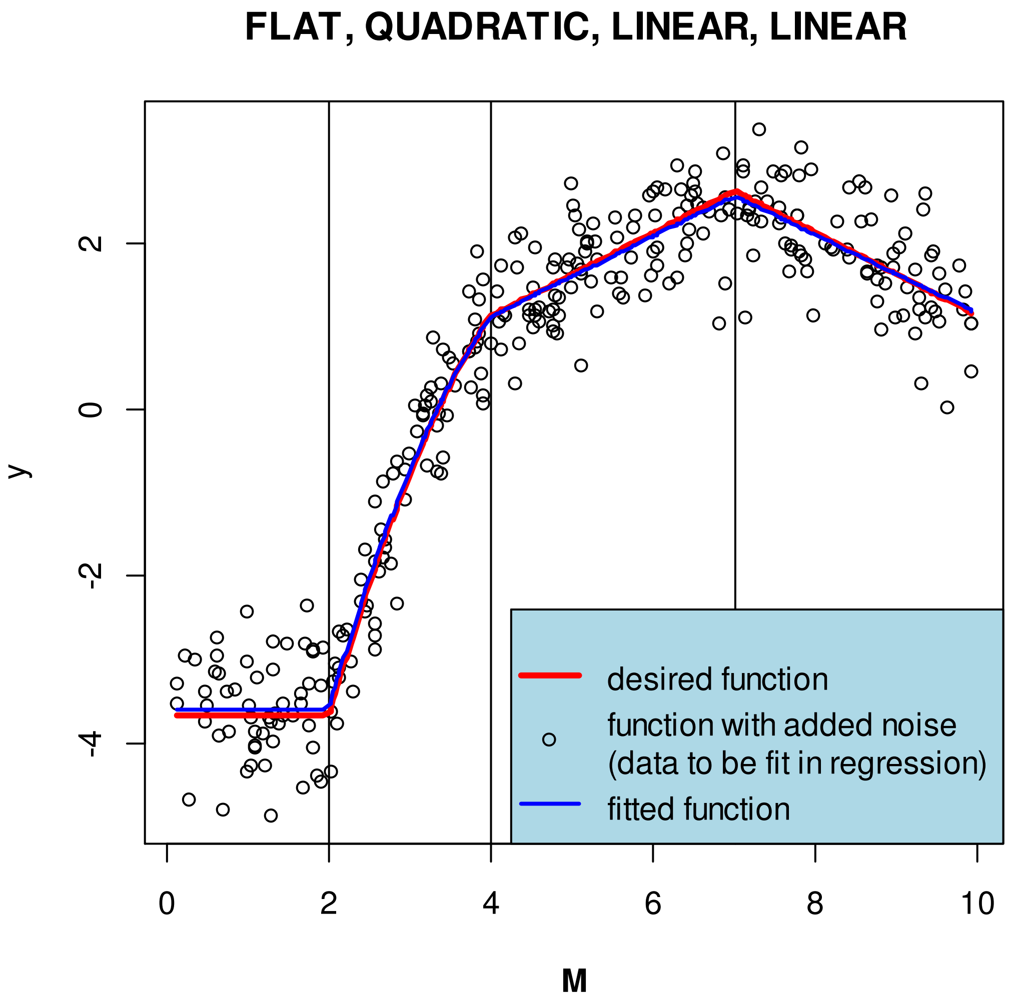
```
# Compute basis functions for each segment (a basis function is not needed
for the first segment, as the intercept
# in the regression will use the data in that segment):
b1.1 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,x-L,R-L))}
b1.2 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,(x-L)^2,(R-L)^2))}
b2 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,x-L,R-L))}
b3 <- function(x,L){ifelse(x<L,0,x-L)}

plot(Msorted, b1.1(Msorted,c[1],c[2]), xlab=substitute(paste(bold("M"))),
ylab="basis function",
  ylim=c(-0.25,8.0), type="l", col = "blue", lwd=3, main=Title)
lines(Msorted, b1.2(Msorted,c[1],c[2]), col = "red", lwd=3)
lines(Msorted, b2(Msorted,c[2],c[3]), col = "green", lwd=3)
lines(Msorted, b3(Msorted,c[3]), col = "purple", lwd=3)
abline(v = c)
```

```
legend(5,8, legend=c("segment 1, basis 1","segment 1, basis 2", "segment 2",
"segment 3"),
        lty=rep(1,4), lwd=rep(3,4), col=c("blue", "red", "green", "purple"),
bg="lightblue")
```

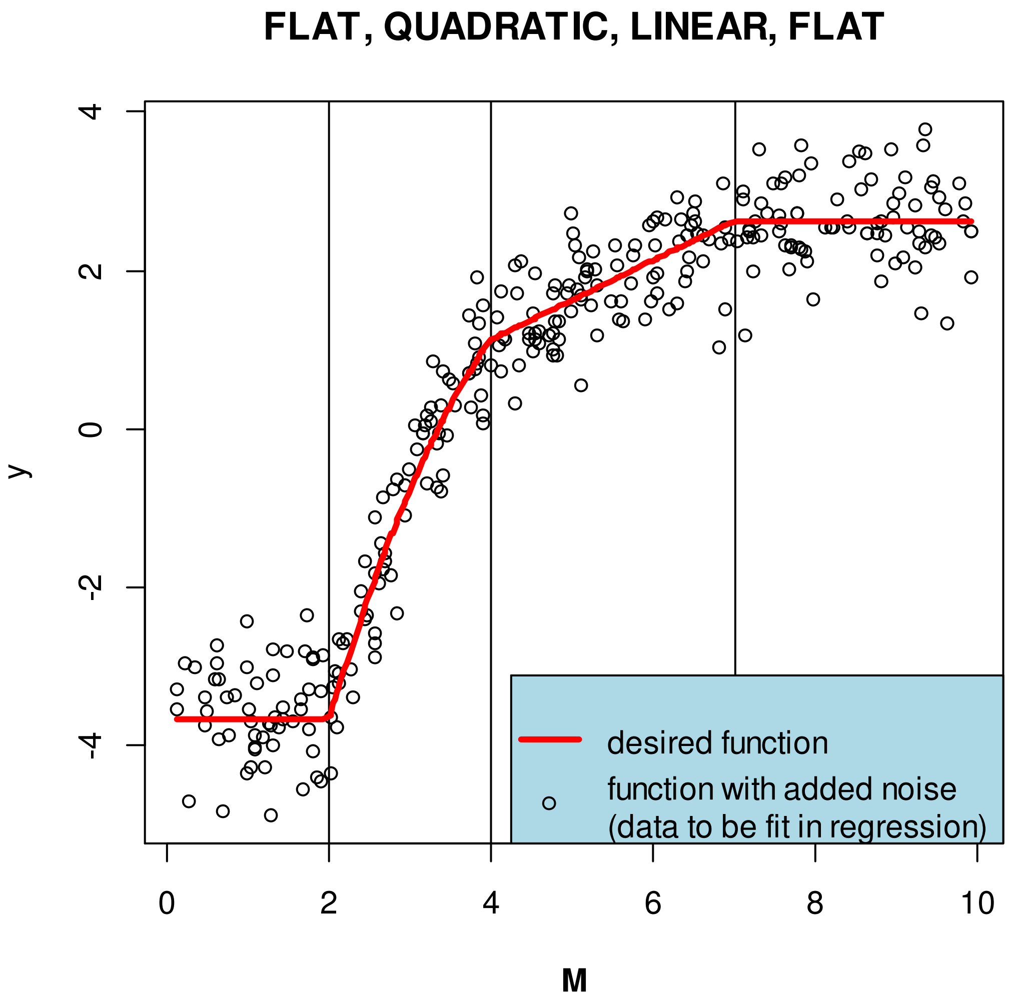


```
Model <- lm(yn ~ b1.1(M,c[1],c[2]) + b1.2(M,c[1],c[2]) + b2(M,c[2],c[3]) +
b3(M,c[3]))

# Regression summary:
summary(Model)

#Call:
#lm(formula = yn ~ b1.1(M, c[1], c[2]) + b1.2(M, c[1], c[2]) +
#    b2(M, c[2], c[3]) + b3(M, c[3]))
#
#Residuals:
#      Min       1Q   Median       3Q      Max
```

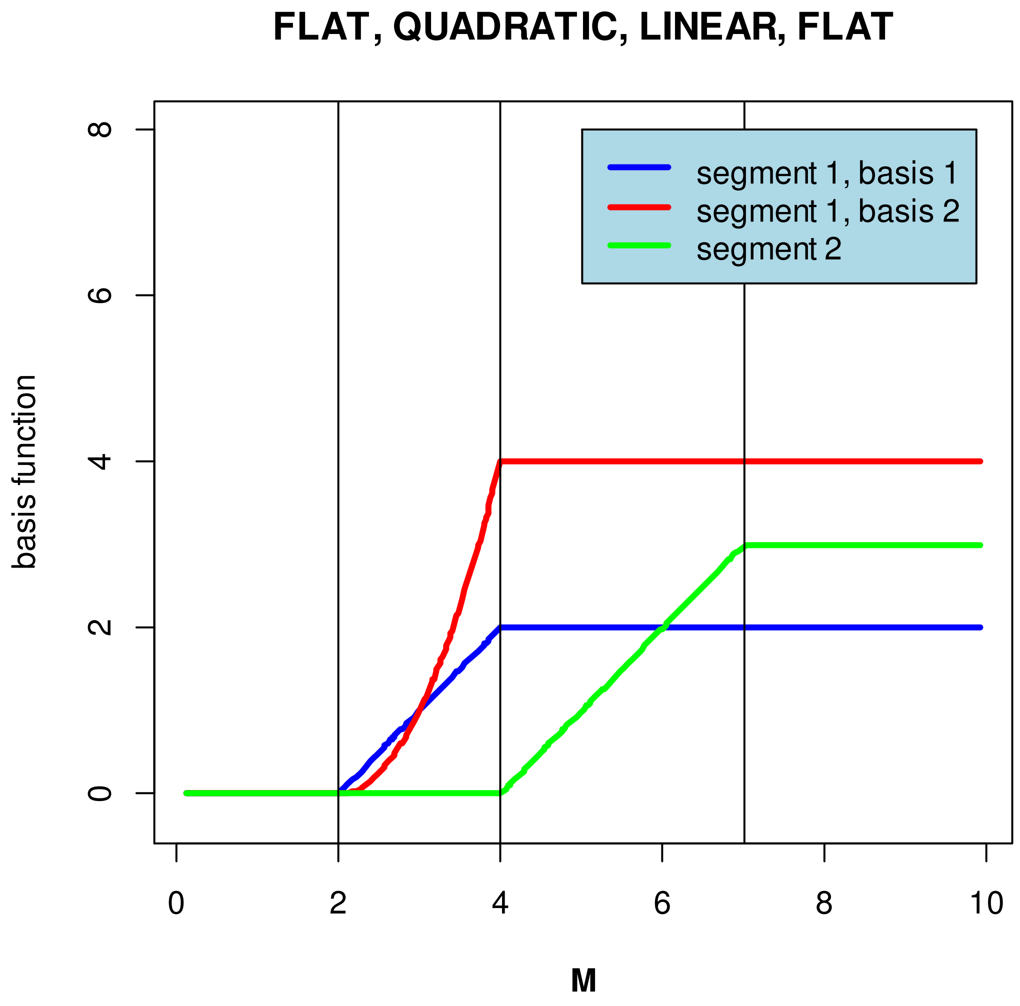
```
#-1.44767 -0.29439 -0.01816  0.34015  1.24359
#
#Coefficients:
#                     Estimate Std. Error t value Pr(>|t|)
#(Intercept)          -3.58901    0.06395 -56.119  < 2e-16 ***
#b1.1(M, c[1], c[2])  3.36741    0.21592  15.596  < 2e-16 ***
#b1.2(M, c[1], c[2]) -0.50595    0.10947  -4.622  5.7e-06 ***
#b2(M, c[2], c[3])    0.48282    0.04407  10.957  < 2e-16 ***
#b3(M, c[3])         -0.46991    0.05168  -9.093  < 2e-16 ***
#---

coef <- coefficients(Model)
#> coef
#        (Intercept) b1.1(M, c[1], c[2]) b1.2(M, c[1], c[2])   b2(M, c[2],
c[3])         b3(M, c[3])
#         -3.5890096           3.3674056          -0.5059541
0.4828228          -0.4699064

# Replot data
plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)
# Plot actual function
abline(v = c)
lines(Msorted,y(Msorted,c), lwd=3, col="red")


# Add predictions
yp <- coef[1]+coef[2]*b1.1(Msorted,c[1], c[2])+coef[3]*b1.2(Msorted,c[1],
      c[2]) +
      coef[4]*b2(Msorted,c[2],c[3]) +
      coef[5]*b3(Msorted,c[3])
lines(Msorted,yp, lwd=2, col="blue")
legend("bottomright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)", "fitted function"),
        lty=c(1,NA,1), lwd=c(3,NA,2), pch=c(NA,1,NA), col=c("red", "black",
"blue"), bg="lightblue")
```


FLAT, QUADRATIC, LINEAR, LINEAR
y
2
0
-2
-4
M
0
2
4
6
8
10
desired function
function with added noise
(data to be fit in regression)
fitted function

```
#------------------------------------------------------------------------------
----------------------
Title <- "FLAT, QUADRATIC, LINEAR, FLAT"
# No basis functions are needed for flat segments
# Define function

# Breakpoints:
c <- c(2, 4, 7)
y <- function(M,c){
  ifelse(M<c[1], -3.67,
    ifelse(M<c[2],1.13+1.4*(M-c[2])-0.5*(M-c[2])^2,
      ifelse(M<=c[3],1.13+0.5*(M-c[2]), 1.13+0.5*(c[3]-c[2]))))
 }


#Generate data
set.seed(1)
n <- 300
M <- runif(n, 0, 10)
Msorted <- sort(M)

# Add some noise:
yn <- y(M,c) + rnorm(n, sd = 0.5)

plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)
abline(v = c)
lines(Msorted, y(Msorted,c),lwd=3, col="red")
legend("bottomright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)"),
        lty=c(1,NA), lwd=c(3,NA), pch=c(NA,1), col=c("red", "black"),
bg="lightblue")
```

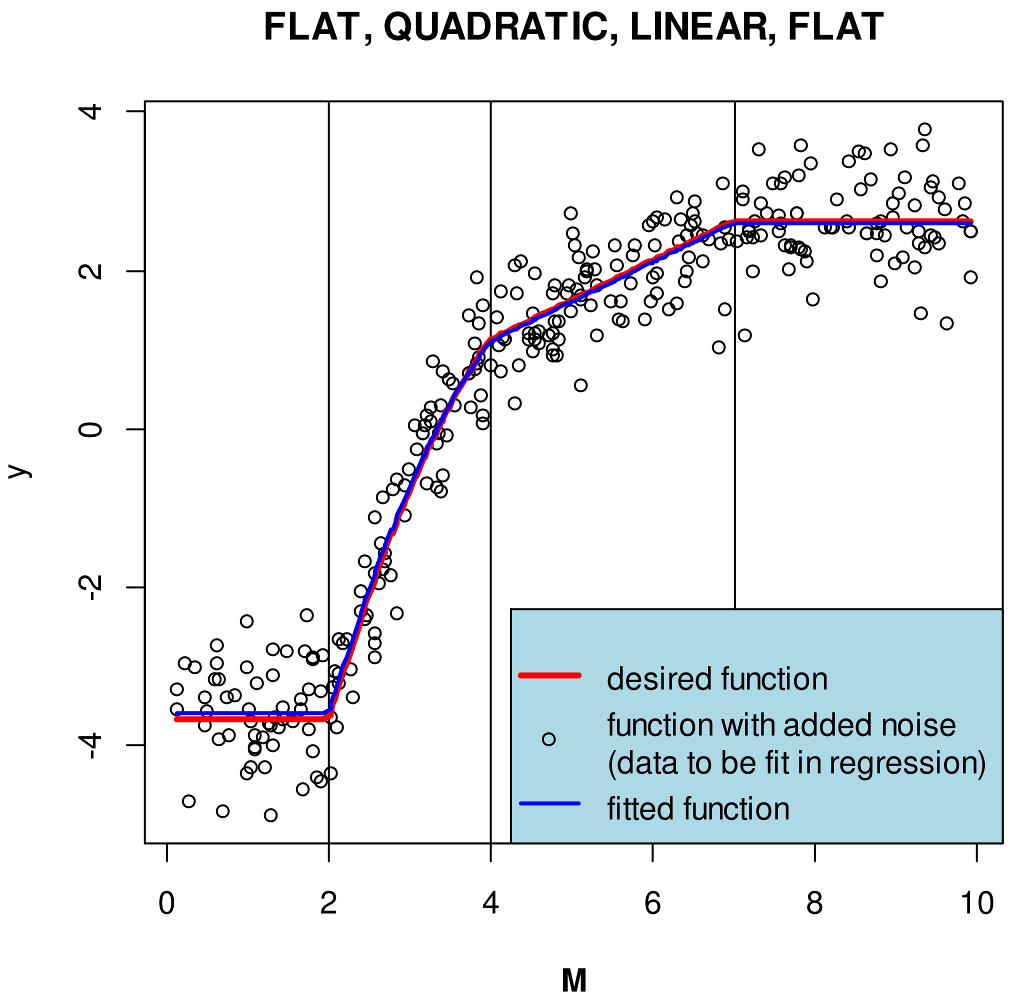


```
# Compute basis function for each region:
b1.1 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,x-L,R-L))}
b1.2 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,(x-L)^2,(R-L)^2))}
b2 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,x-L,R-L))}

plot(Msorted, b1.1(Msorted,c[1],c[2]), xlab=substitute(paste(bold("M"))),
ylab="basis function",
  ylim=c(-0.25,8.0), type="l", col = "blue", lwd=3, main=Title)
lines(Msorted, b1.2(Msorted,c[1],c[2]), col = "red", lwd=3)
lines(Msorted, b2(Msorted,c[2],c[3]), col = "green", lwd=3)
abline(v = c)
legend(5,8, legend=c("segment 1, basis 1","segment 1, basis 2", "segment 2"),
        lty=rep(1,3), lwd=rep(3,3), col=c("blue", "red", "green"),
bg="lightblue")
```



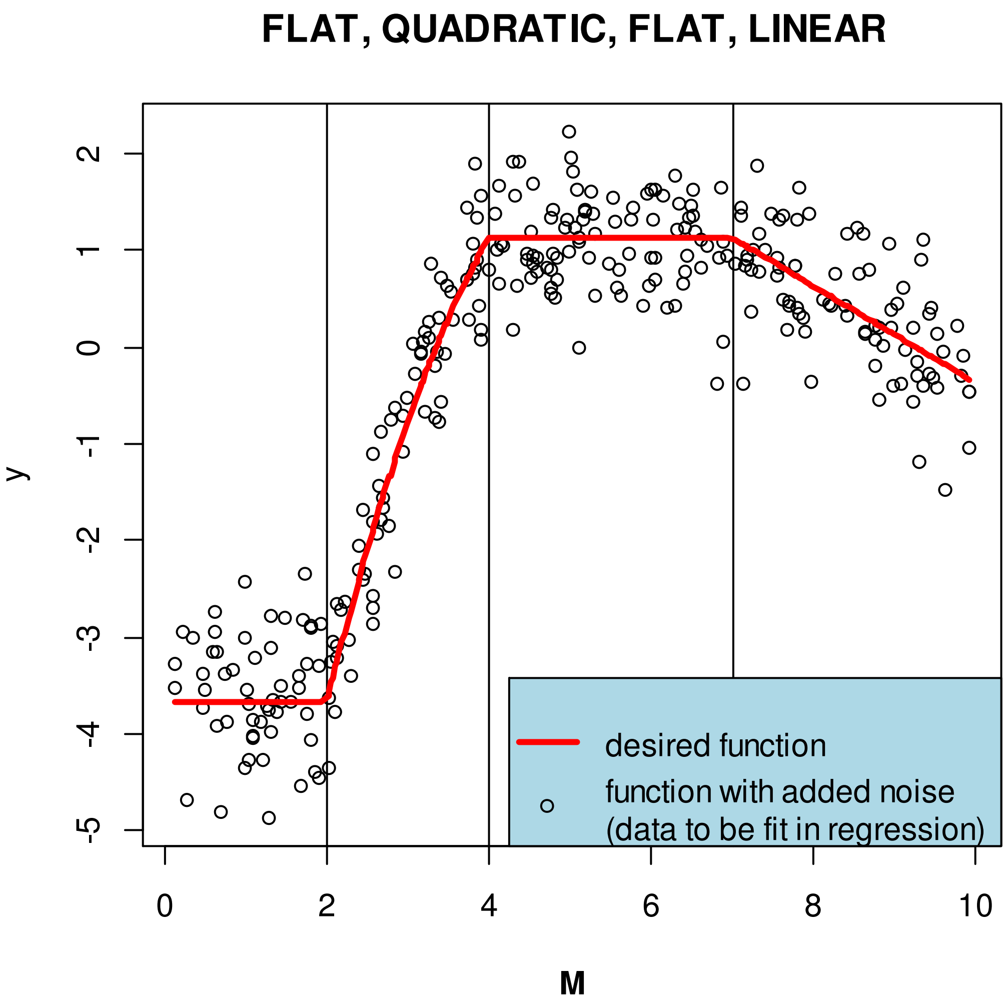
```
Model <- lm(yn ~ b1.1(M,c[1],c[2]) + b1.2(M,c[1],c[2]) + b2(M,c[2],c[3]) )
# Regression summary:
summary(Model)

#Call:
#lm(formula = yn ~ b2(M, c[1], c[2]) + b2.2(M, c[1], c[2]) + b3(M,
#    c[2], c[3]))
#
#Residuals:
#     Min       1Q   Median       3Q      Max
#-1.47883 -0.29232 -0.01076  0.33950  1.24418
#
#Coefficients:
#                    Estimate Std. Error t value Pr(>|t|)
#(Intercept)         -3.58959    0.06387 -56.198  < 2e-16 ***
#b2(M, c[1], c[2])    3.37824    0.21487  15.722  < 2e-16 ***
```

```
#b2.2(M, c[1], c[2]) -0.51440    0.10839  -4.746 3.24e-06 ***
#b3(M, c[2], c[3])    0.49841    0.03497  14.253  < 2e-16 ***
#---

coef <- coefficients(Model)
#> coef
#        (Intercept)   b2(M, c[1], c[2]) b2.2(M, c[1], c[2])   b3(M, c[2],
c[3])
#         -3.5895926           3.3782354          -0.5143986
0.4984074

# Replot data
plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)
# Plot actual function
abline(v = c)
lines(Msorted,y(Msorted,c), lwd=3, col="red")


# Add predictions
yp <- coef[1]+coef[2]*b1.1(Msorted,c[1], c[2])+coef[3]*b1.2(Msorted,c[1],
c[2]) +
      coef[4]*b2(Msorted,c[2],c[3])
lines(Msorted,yp, lwd=2, col="blue")
legend("bottomright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)", "fitted function"),
        lty=c(1,NA,1), lwd=c(3,NA,2), pch=c(NA,1,NA), col=c("red", "black",
"blue"), bg="lightblue")
```


FLAT, QUADRATIC, LINEAR, FLAT
y
4
2
0
-2
-4
M
0
2
4
6
8
10
desired function
function with added noise
(data to be fit in regression)
fitted function

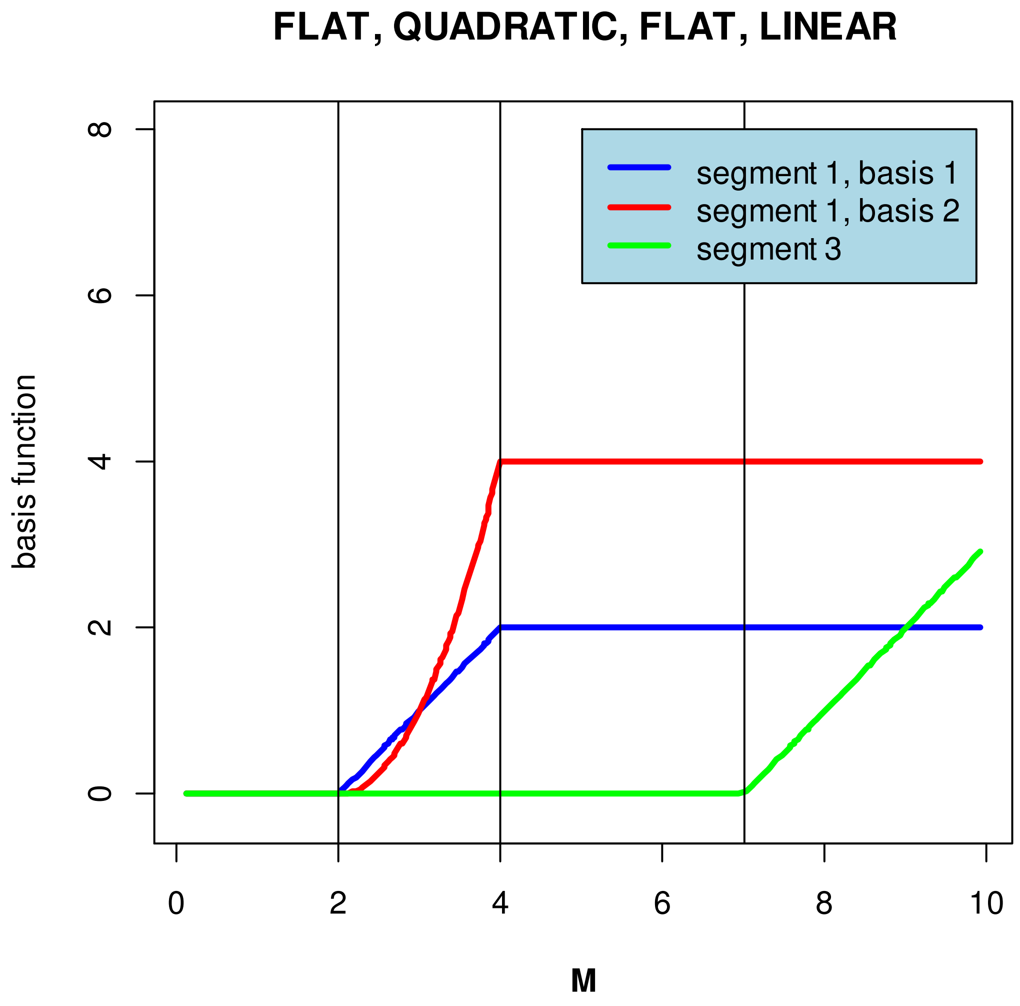
```
#-----------------------------------------------------------------------------
----------------------

Title <- "FLAT, QUADRATIC, FLAT, LINEAR"
# No basis functions are needed for flat segments#
# Define function


# Breakpoints:
c <- c(2, 4, 7)
y <- function(M,c){
  ifelse(M<c[1], -3.67,
    ifelse(M<c[2],1.13+1.4*(M-c[2])-0.5*(M-c[2])^2,
      ifelse(M<=c[3],1.13, 1.13-0.5*(M-c[3]))))
 }



#Generate data
set.seed(1)
n <- 300
M <- runif(n, 0, 10)
Msorted <- sort(M)


# Add some noise:
yn <- y(M,c) + rnorm(n, sd = 0.5)


plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)
abline(v = c)
lines(Msorted, y(Msorted,c),lwd=3, col="red")
```

```
legend("bottomright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)"),

        lty=c(1,NA), lwd=c(3,NA), pch=c(NA,1), col=c("red", "black"),
bg="lightblue")
```



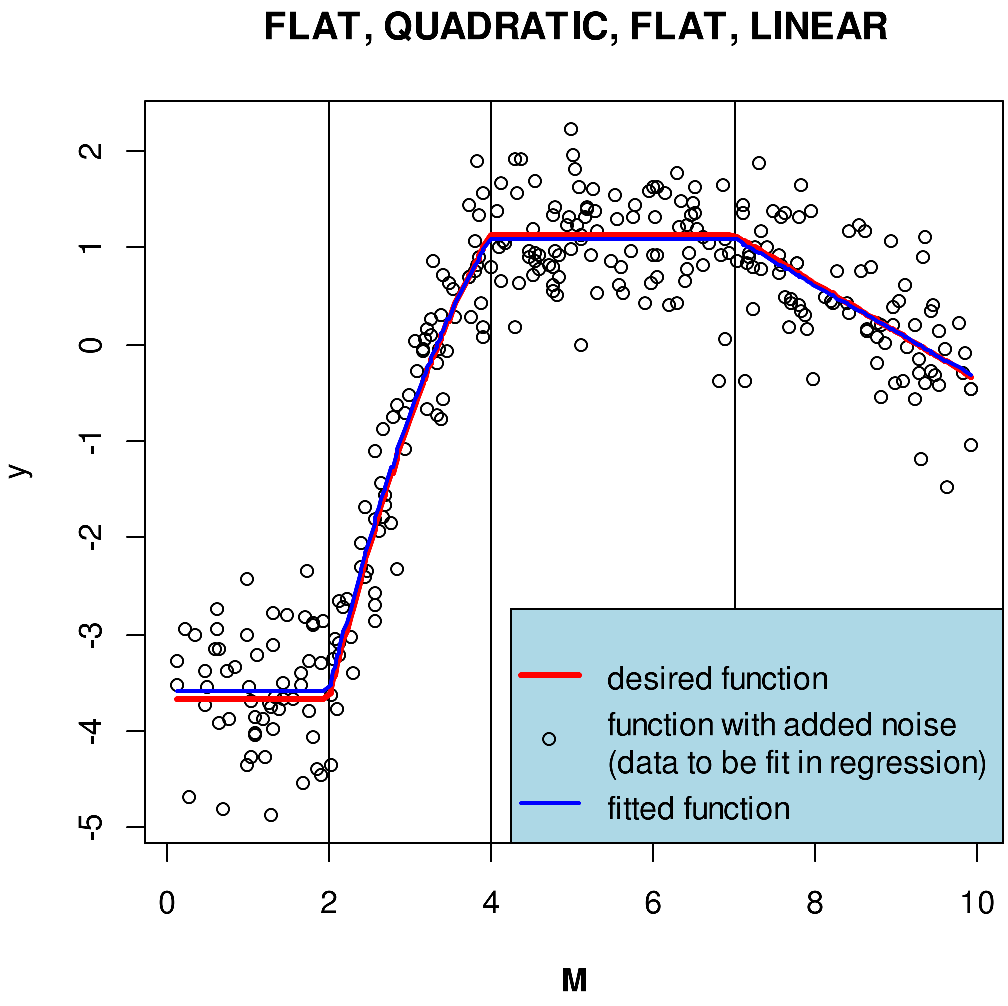
```
# Compute basis function for each region:

b1.1 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,x-L,R-L))}

b1.2 <- function(x,L,R){ifelse(x<L,0,ifelse(x<R,(x-L)^2,(R-L)^2))}

b3 <- function(x,L){ifelse(x<L,0,x-L)}
```

```
plot(Msorted, b1.1(Msorted,c[1],c[2]), xlab=substitute(paste(bold("M"))),
ylab="basis function",

  ylim=c(-0.25,8.0), type="l", col = "blue", lwd=3, main=Title)

lines(Msorted, b1.2(Msorted,c[1],c[2]), col = "red", lwd=3)

lines(Msorted, b3(Msorted,c[3]), col = "green", lwd=3)

abline(v = c)

legend(5,8, legend=c("segment 1, basis 1","segment 1, basis 2", "segment 3"),

        lty=rep(1,3), lwd=rep(3,3), col=c("blue", "red", "green"),
bg="lightblue")
```



```
Model <- lm(yn ~ b1.1(M,c[1],c[2]) + b1.2(M,c[1],c[2]) + b3(M,c[3]) )
# Regression summary:
summary(Model)


#Call:
#lm(formula = yn ~ b1.1(M, c[1], c[2]) + b1.2(M, c[1], c[2]) +
#    b3(M, c[3]))
#
#Residuals:
#     Min       1Q   Median       3Q      Max
#-1.46848 -0.29279 -0.01513  0.34001  1.24486
#
#Coefficients:
#                    Estimate Std. Error t value Pr(>|t|)
#(Intercept)         -3.59027    0.06378 -56.292  < 2e-16 ***
#b1.1(M, c[1], c[2])  3.39088    0.20705  16.377  < 2e-16 ***
#b1.2(M, c[1], c[2]) -0.52426    0.09875  -5.309 2.17e-07 ***
#b3(M, c[3])         -0.48214    0.04100 -11.761  < 2e-16 ***
#---
#Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#
#Residual standard error: 0.5123 on 296 degrees of freedom
#Multiple R-squared:  0.9271,    Adjusted R-squared:  0.9264
#F-statistic:  1256 on 3 and 296 DF,  p-value: < 2.2e-16#Call:


coef <- coefficients(Model)
> coef
```

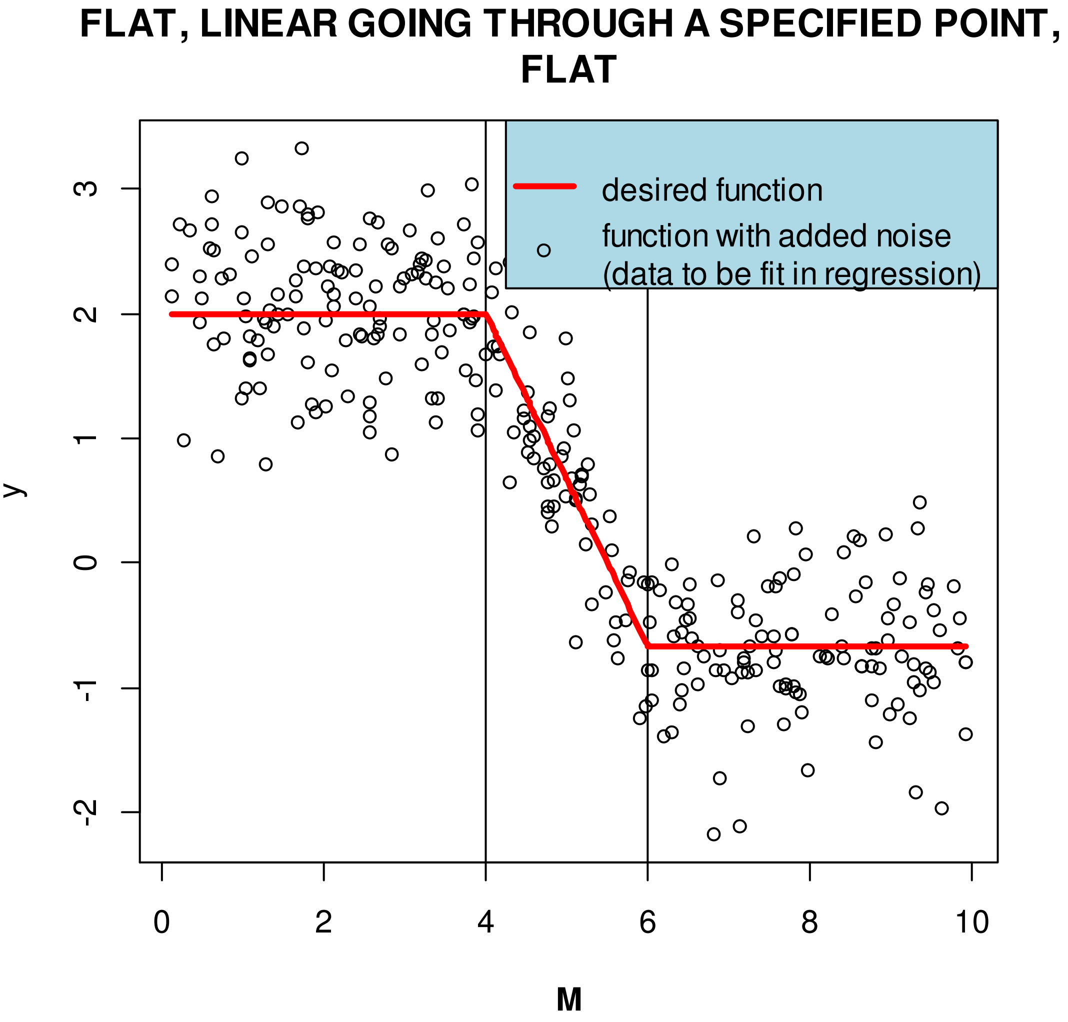
```
        (Intercept) b1.1(M, c[1], c[2]) b1.2(M, c[1], c[2])         b3(M,
c[3])

         -3.5902734           3.3908799          -0.5242581           -
0.4821406



# Replot data

plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main=Title)

# Plot actual function

abline(v = c)

lines(Msorted,y(Msorted,c), lwd=3, col="red")




# Add predictions

yp <- coef[1]+coef[2]*b1.1(Msorted,c[1], c[2])+coef[3]*b1.2(Msorted,c[1],
c[2]) +

      coef[4]*b3(Msorted,c[3])

lines(Msorted,yp, lwd=2, col="blue")

legend("bottomright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)", "fitted function"),

        lty=c(1,NA,1), lwd=c(3,NA,2), pch=c(NA,1,NA), col=c("red", "black",
"blue"), bg="lightblue")
```


FLAT, QUADRATIC, FLAT, LINEAR
y
2
1
0
-1
-2
-3
-4
-5
0
2
4
6
8
10
M
desired function
function with added noise
(data to be fit in regression)
fitted function

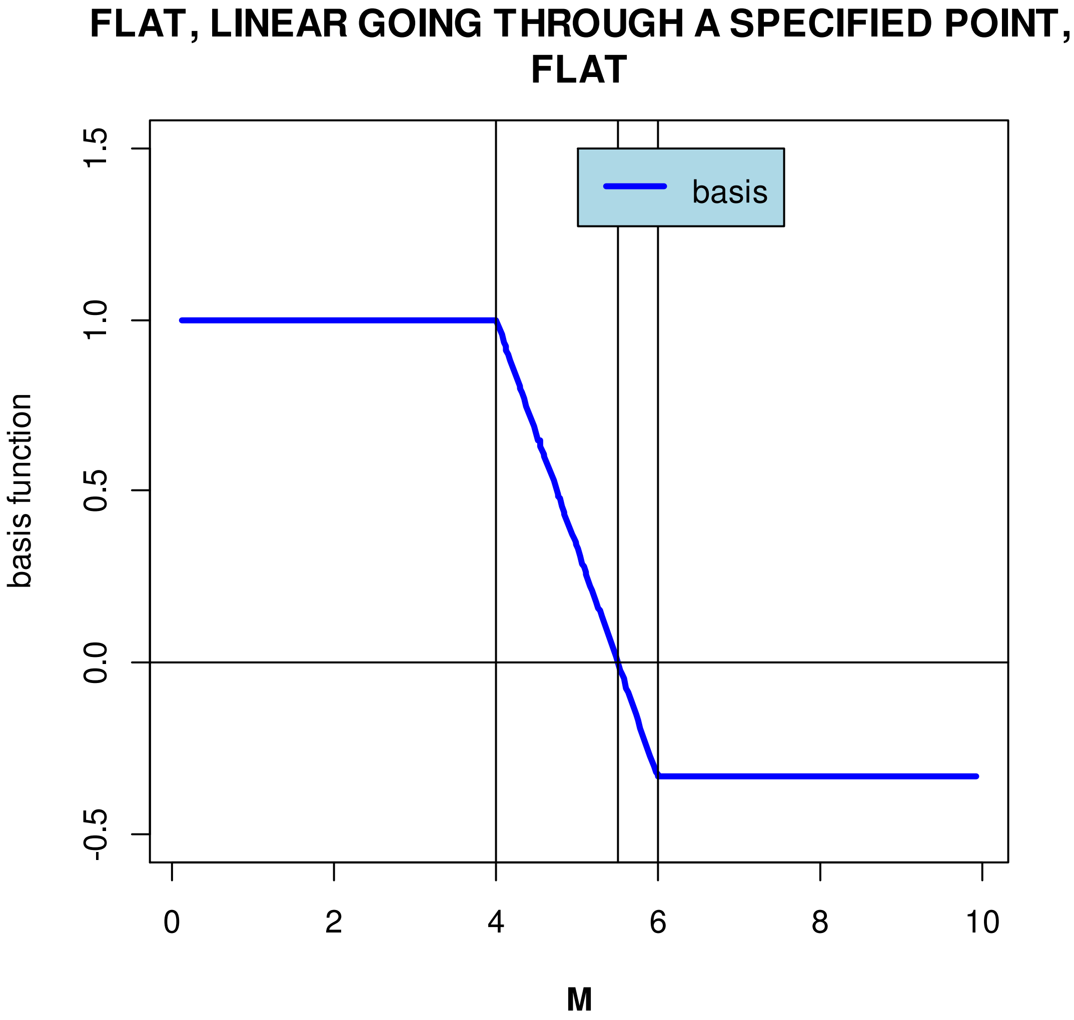
FLAT, LINE CROSSING 0.0 AT XC, FLAT

How should a model that is flat to c[1], forced to cross 0.0 at x = xc, and then is flat beyond c[2] be specified? Thinking about it, there is only one basis function, even though it has a break at x=c[1]. The reason is that the slope of the line between c[1] and c[2] is determined by the value of constant portion for x < c[1] and the condition that the line crosses the zero line at x=xc. Therefore, the slope is NOT a regression parameter. There is only one regression parameter.

```
#FLAT, LINEAR GOING THROUGH A SPECIFIED POINT, FLAT

# Define function

# Breakpoints:
c <- c(4, 6)
cz <- 5.5
slope <- (2.0-0.0)/(cz-c[1])
y <- function(x,c){
  ifelse(x<c[1], 2.0,
    ifelse(x<c[2],2.0 - slope*(x-c[1]), 2.0 - slope*(c[2]-c[1]) ))
 }


#Generate data
set.seed(1)
n <- 300
M <- runif(n, 0, 10)
Msorted <- sort(M)

# Add some noise:
yn <- y(M,c) + rnorm(n, sd = 0.5)

plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main="FLAT, LINEAR GOING THROUGH A SPECIFIED POINT,\nFLAT")
abline(v = c)
lines(Msorted, y(Msorted,c),lwd=3, col="red")
legend("topright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)"),
        lty=c(1,NA), lwd=c(3,NA), pch=c(NA,1), col=c("red", "black"),
bg="lightblue")
```

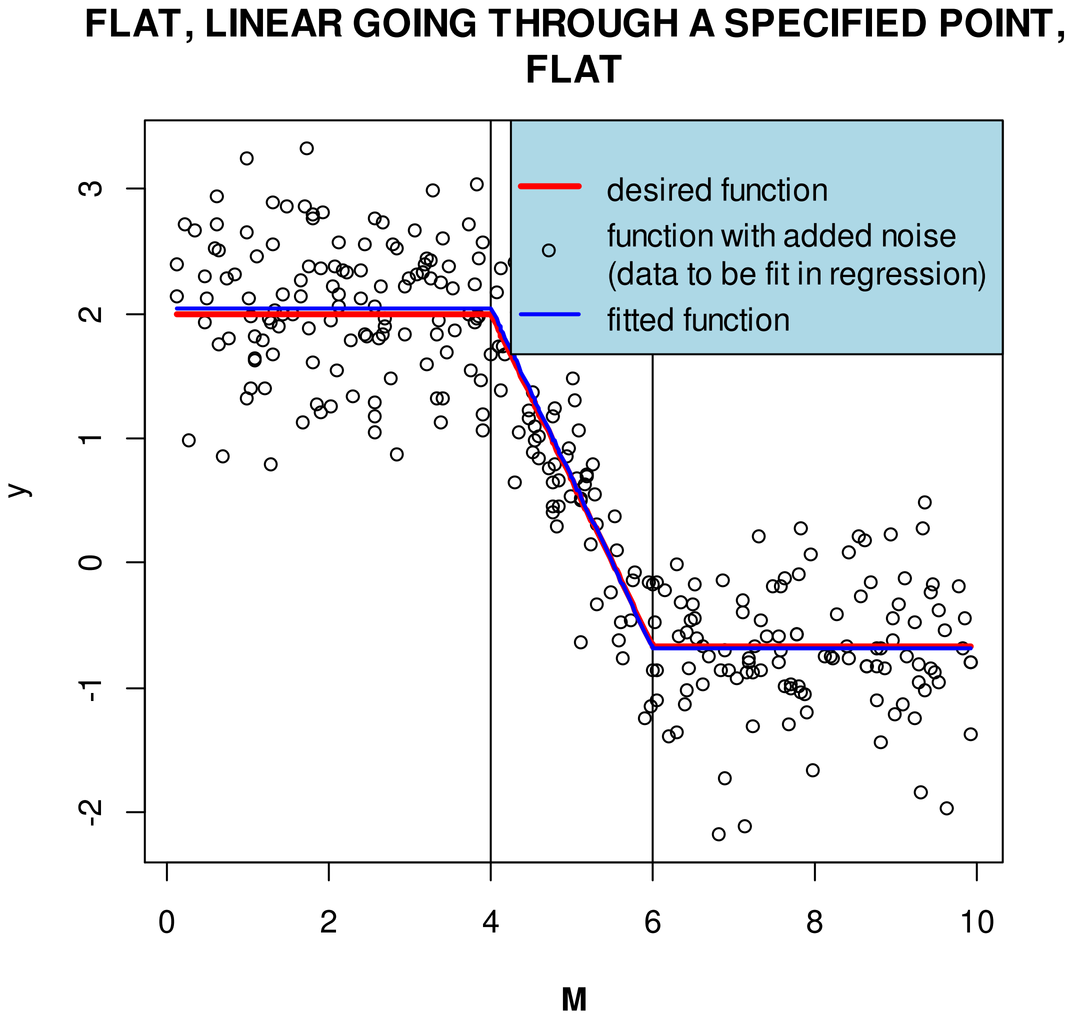


```
b1 <- function(x,L,xc,R){ifelse(x<L,1,ifelse(x<R,1-(x-L)/(xc-L),1-(R-L)/(xc-
L)))}

plot(Msorted, b1(Msorted,c[1],cz,c[2]), xlab=substitute(paste(bold("M"))),
ylab="basis function",
  ylim=c(-0.5,1.5), type="l", col = "blue", lwd=3, main="FLAT, LINEAR GOING
THROUGH A SPECIFIED POINT,\nFLAT”)
abline(h=0)
abline(v = c(c[1],cz,c[2]))
legend(5,1.5, legend=c("basis"),
        lty=rep(1,1), lwd=rep(3,1), col=c("blue"), bg="lightblue")
```



```
Model <- lm(yn ~ -1 + b1(M,c[1],cz,c[2]) )
# Regression summary:
summary(Model)

#Call:
#lm(formula = yn ~ -1 + b1(M, c[1], cz, c[2]))
#
#Residuals:
#     Min       1Q   Median       3Q      Max
#-1.48759 -0.30188 -0.03717  0.33624  1.27527
#
#Coefficients:
#                      Estimate Std. Error t value Pr(>|t|)
#b1(M, c[1], cz, c[2])   2.0493     0.0411   49.86   <2e-16 ***
#---
```

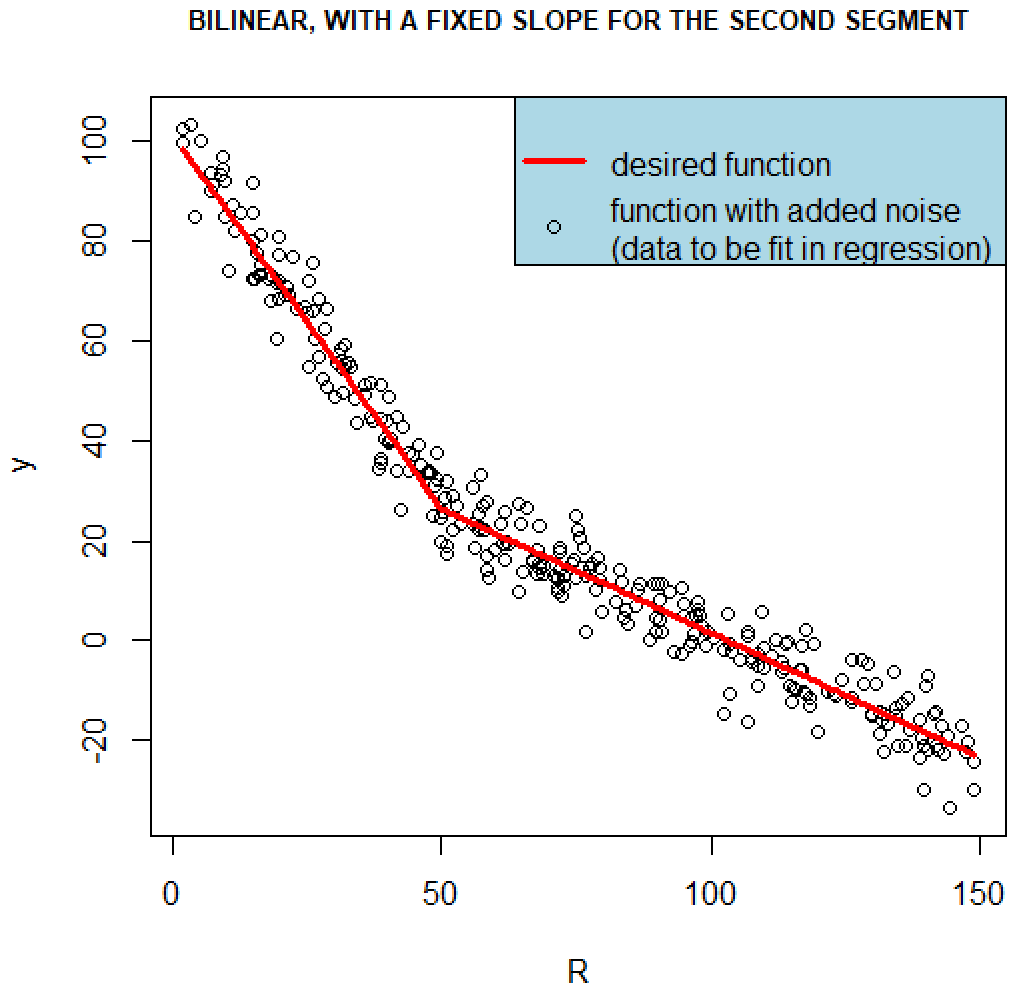
```
#Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
#
#Residual standard error: 0.5107 on 299 degrees of freedom
#Multiple R-squared:  0.8927,    Adjusted R-squared:  0.8923
#F-statistic:  2486 on 1 and 299 DF,  p-value: < 2.2e-16

coef <- coefficients(Model)
coef
#b1(M, c[1], cz, c[2])
#              2.04931

# Replot data
plot(M, yn,  xlab=substitute(paste(bold("M"))), ylab="y", col="black",
main="FLAT, LINEAR GOING THROUGH A SPECIFIED POINT,\nFLAT")
# Plot actual function
abline(v = c)
lines(Msorted,y(Msorted,c), lwd=3, col="red")


# Add predictions
yp <- coef[1]*b1(Msorted,c[1],cz,c[2])
lines(Msorted,yp, lwd=2, col="blue")
legend("topright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)", "fitted function"),
        lty=c(1,NA,1), lwd=c(3,NA,2), pch=c(NA,1,NA), col=c("red", "black",
"blue"), bg="lightblue")
```


FLAT, LINEAR GOING THROUGH A SPECIFIED POINT, FLAT
desired function
function with added noise (data to be fit in regression)
fitted function
y
M
-2
-1
0
1
2
3
0
2
4
6
8
10

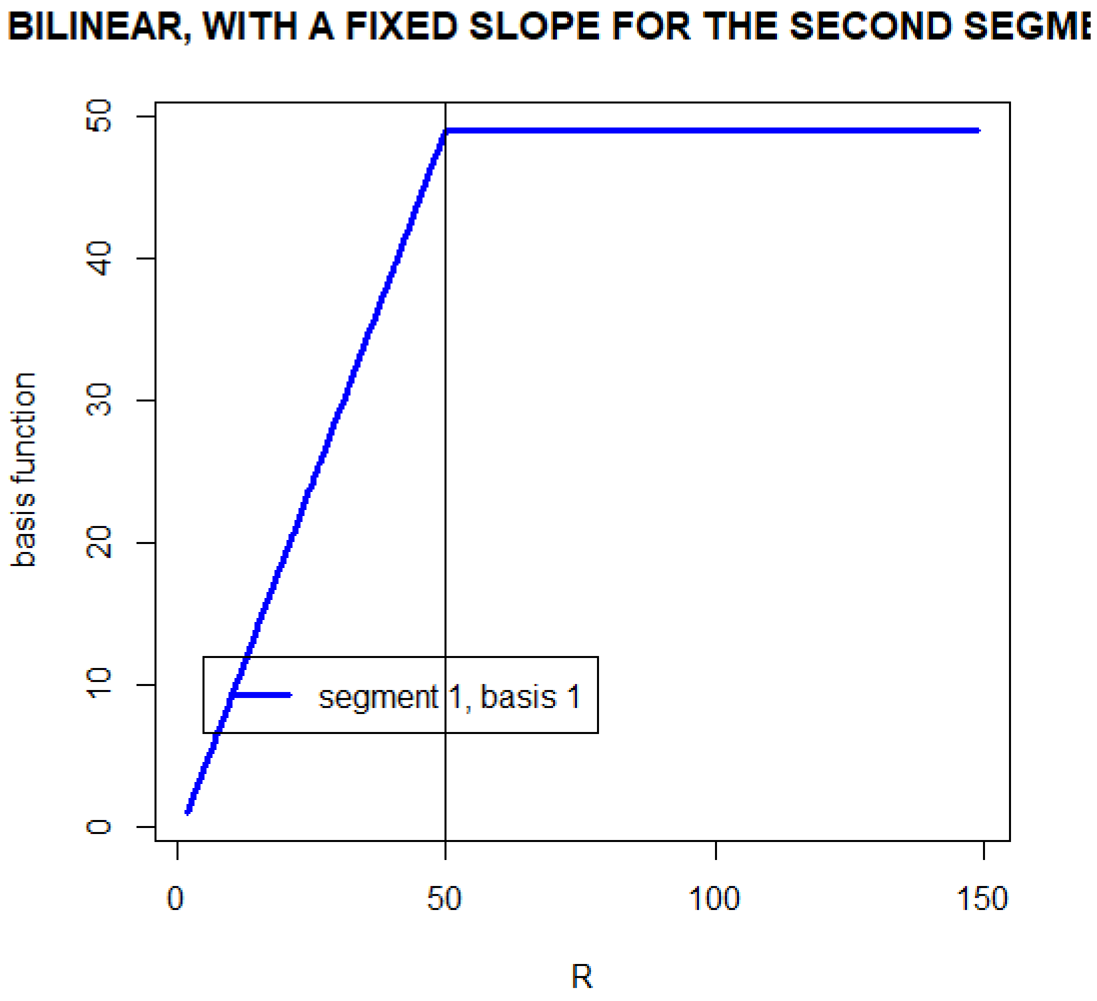
BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGMENT

This example requires the basis function for the second segment to be defined and thus it needs to be subtracted from the "data" before the regression. The coefficient for this segment is then added to the regression coefficient to get the final fitted function.

```
Title <- "BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGMENT"

# Define function
y <- function(R,YatRref,s1, Rh, s2){
  ifelse(R <= Rh, YatRref + s1*(R-Rref), YatRref + s1*(Rh-Rref) + s2*(R-Rh))
}

# Breakpoints:
YatRref <- 100
Rref <- 1.0
Rh <- 50
s1 <- -1.5
s2 <- -0.5

#Generate data (a set without the second segment
set.seed(1)
n <- 300
R <- runif(n, 0, 150)
Rsorted <- sort(R)

# Add some noise:
yn <- y(R,YatRref,s1, Rh, s2) + rnorm(n, sd = 5.0)

plot(R, yn,  xlab="R", ylab="y", col="black")
lines(Rsorted, y(Rsorted, YatRref,s1, Rh, s2),lwd=3, col="red")
title(main=Title, cex.main=0.8)
legend("topright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)"),
        lty=c(1,NA), lwd=c(3,NA), pch=c(NA,1), col=c("red", "black"),
bg="lightblue")
```

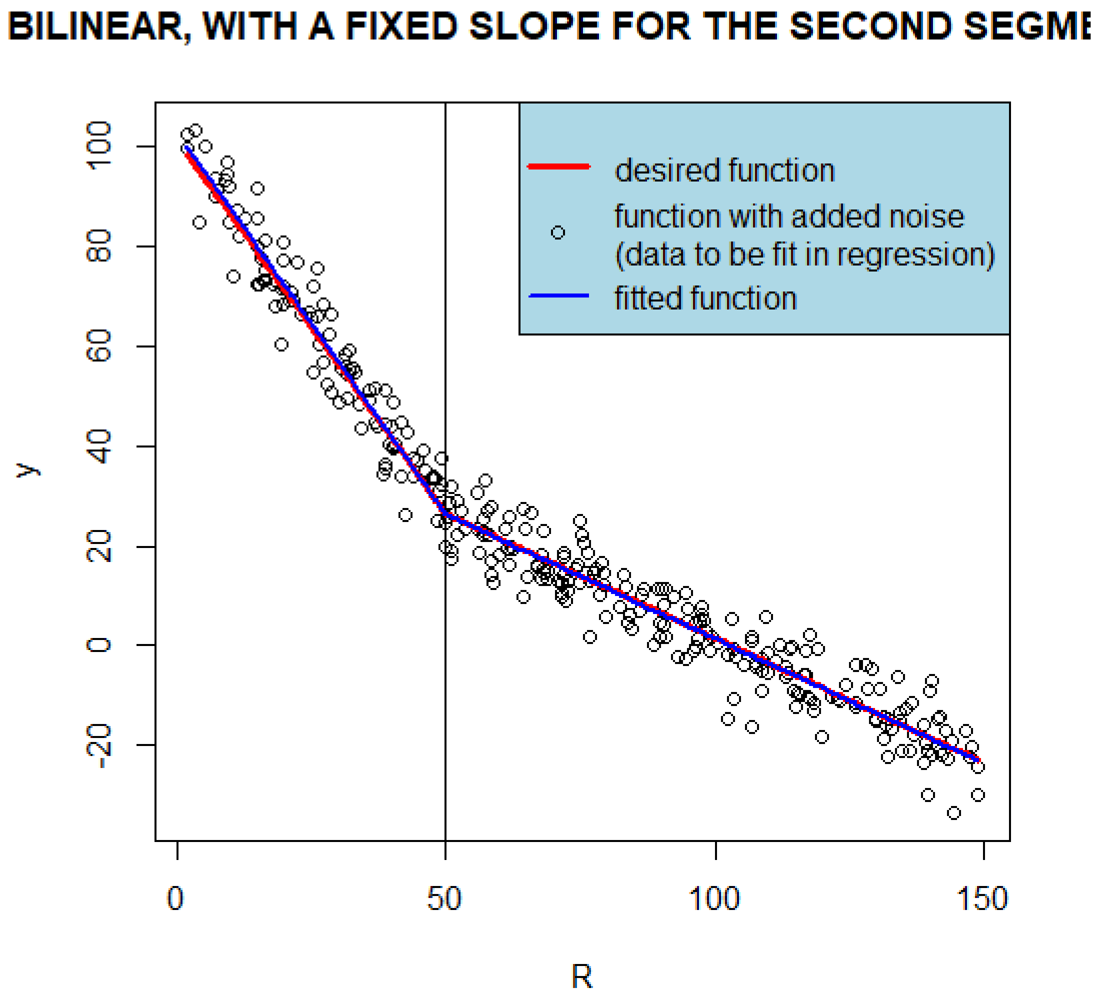


```
# Define basis function for each region (a single line)
b1 <- function(x, Rref, Rh){ifelse(x<Rh,x-Rref,Rh-Rref)}

plot(Rsorted, b1(Rsorted,Rref,Rh), xlab="R", ylab="basis function",
  ylim=range(b1(Rsorted,Rref,Rh)), type="l", col = "blue", lwd=3, main=Title)
abline(v = Rh)
legend(5,12, legend=c("segment 1, basis 1"),
        lty=rep(1,1), lwd=rep(3,1), col=c("blue"))
```

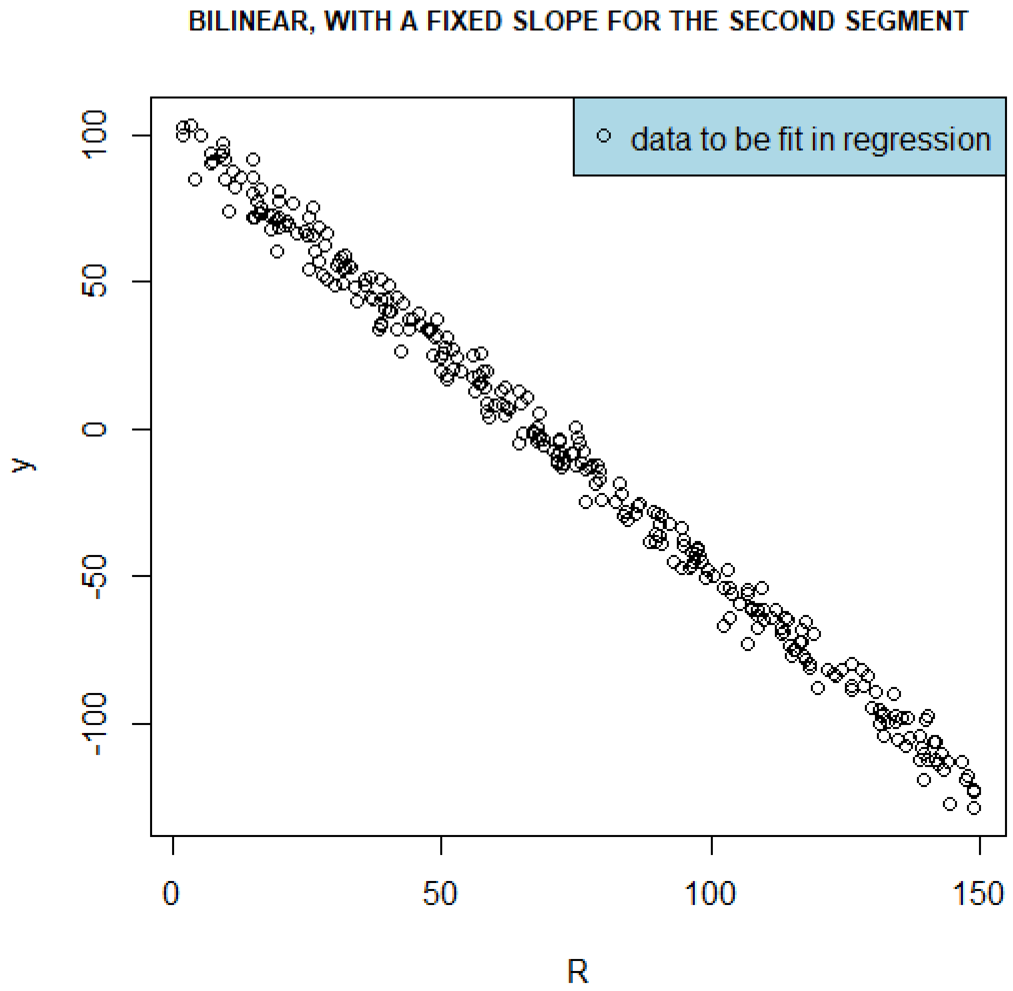


```
yn4lm <- yn - s2*(R-Rref)

Model <- lm(yn4lm ~ b1(R,Rref,Rh) )
# Regression summary:
summary(Model)

#Call:
#lm(formula = yn4lm ~ b1(R, Rref, Rh))
#
#Residuals:
#     Min       1Q   Median       3Q      Max 
#-14.8869  -3.0270  -0.1101   3.4391  12.6031 
#
#Coefficients:
#                  Estimate Std. Error t value Pr(>|t|)    
#(Intercept)      101.47308    0.97651  103.91   <2e-16 ***
#b1(R, Rref, Rh)   -1.03319    0.02248  -45.97   <2e-16 ***
```

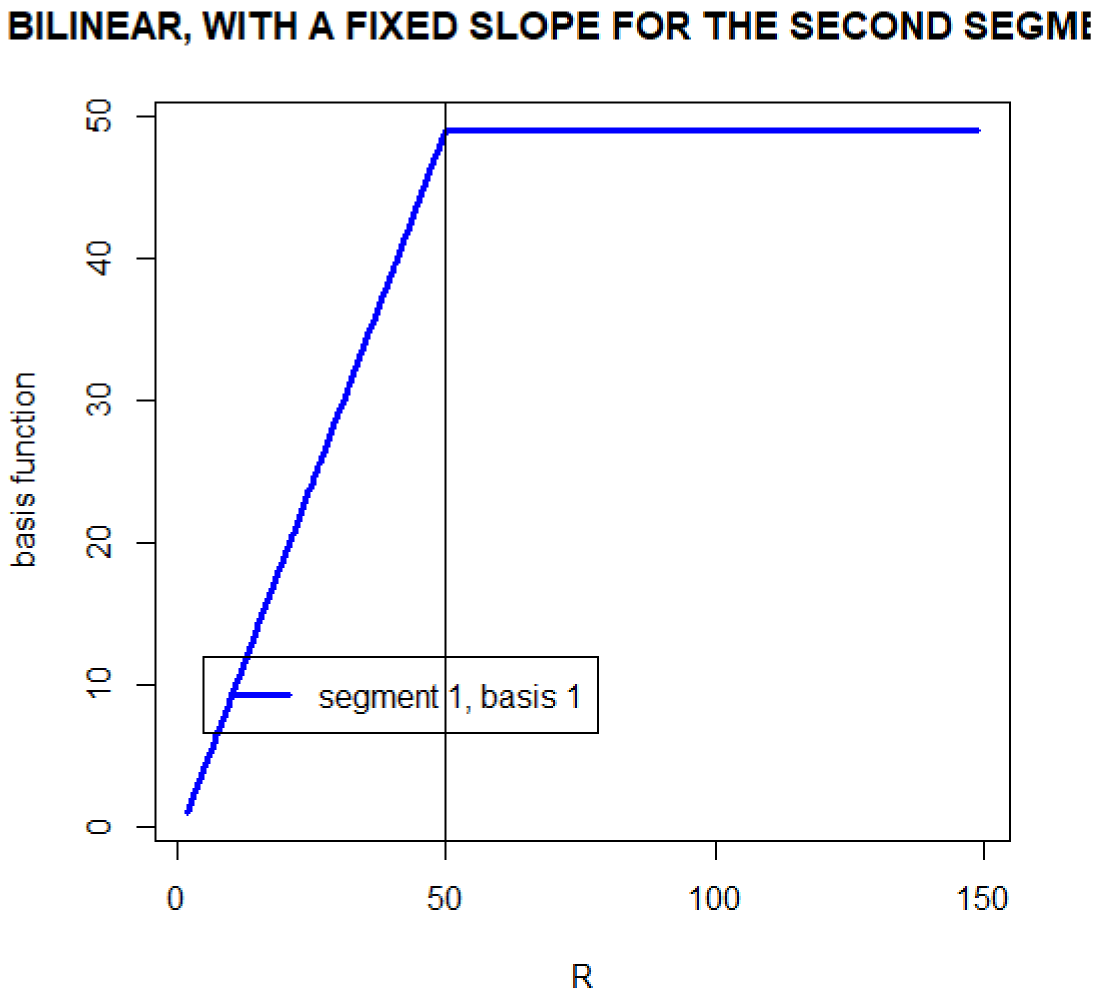
```
#---
#Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
#
#Residual standard error: 5.108 on 298 degrees of freedom
#Multiple R-squared:  0.8764,    Adjusted R-squared:  0.876
#F-statistic:  2113 on 1 and 298 DF,  p-value: < 2.2e-16

coef <- coefficients(Model)
#> coef
#    (Intercept) b1(R, Rref, Rh)
#     101.473080       -1.033192

# Replot data
plot(R, yn,  xlab="R", ylab="y", col="black", main=Title)
# Plot actual function
abline(v = Rh)
lines(Rsorted, y(Rsorted, YatRref,s1, Rh, s2),lwd=3, col="red")

# Add predictions

#Add constrained basis back, and redefine coefficients by factoring out
common distance term for first line:
#yp <- coef[1] + (coef[2])*b1(R,Rref,Rh) + s2*(Rsorted-Rref)
#but for first line (R<Rh), b1 <-  R-Rref, so
#yp <- coef[1] + (coef[2])*(R-Rref) + s2*(R-Rref)
#and
#yp <- coef[1] + (coef[2] + s2)*(R-Rref)
#so
YatRrefBilinear <- coef[1]
s1Bilinear <- coef[2] + s2
s2Bilinear <- s2
#and
yp <- ifelse(Rsorted < Rh, YatRrefBilinear + s1Bilinear*(Rsorted-Rref),
YatRrefBilinear + s1Bilinear*(Rh-Rref) + s2Bilinear*(Rsorted-Rh))
lines(Rsorted,yp, lwd=2, col="blue")
legend("topright", legend=c("desired function", "function with added
noise\n(data to be fit in regression)", "fitted function"),
        lty=c(1,NA,1), lwd=c(3,NA,2), pch=c(NA,1,NA), col=c("red", "black",
"blue"), bg="lightblue")
```


BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGME
desired function
function with added noise
(data to be fit in regression)
fitted function
y
100
80
60
40
20
0
-20
0
50
100
150
R

```
#$$$$$$$$$$$$$$$$$$$ NOW TRY TO FIT DATA WITH A DIFFERENT SLOPE FOR THE
SECOND SEGMENT THAN THE ASSUMED SLOPE

Title <- "BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGMENT"

# Define function
y <- function(R,YatRref,s1, Rh, s2){
  ifelse(R <= Rh, YatRref + s1*(R-Rref), YatRref + s1*(Rh-Rref) + s2*(R-Rh))
}

# Breakpoints:
YatRref <- 100
Rref <- 1.0
Rh <- 50
s1 <- -1.5
s2Data <- -1.5
s2 <- -0.5

#Generate data (a set without the second segment
set.seed(1)
n <- 300
R <- runif(n, 0, 150)
Rsorted <- sort(R)

# Add some noise:
yn <- y(R,YatRref,s1, Rh, s2Data) + rnorm(n, sd = 5.0)

plot(R, yn,  xlab="R", ylab="y", col="black")
title(main=Title, cex.main=0.8)
legend("topright", legend=c("data to be fit in regression"),
        pch=c(1), col=c("black"), bg="lightblue")
```

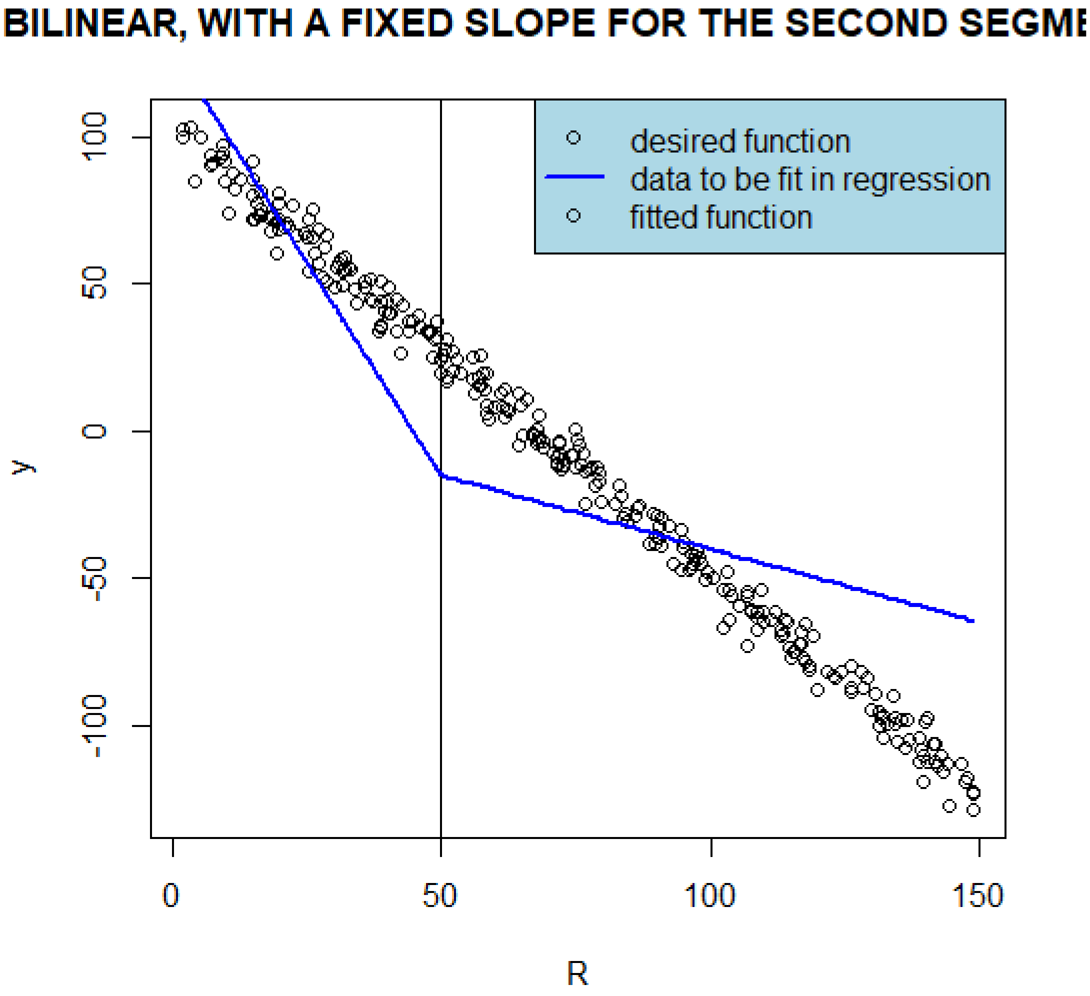


```
# Define basis function for each region (a single line)
b1 <- function(x, Rref, Rh){ifelse(x<Rh,x-Rref,Rh-Rref)}

plot(Rsorted, b1(Rsorted,Rref,Rh), xlab="R", ylab="basis function",
  ylim=range(b1(Rsorted,Rref,Rh)), type="l", col = "blue", lwd=3, main=Title)
abline(v = Rh)
legend(5,12, legend=c("segment 1, basis 1"),
        lty=rep(1,1), lwd=rep(3,1), col=c("blue"))
```



```
yn4lm <- yn - s2*(R-Rref)

Model <- lm(yn4lm ~ b1(R,Rref,Rh) )
# Regression summary:
summary(Model)

#Call:
#lm(formula = yn4lm ~ b1(R, Rref, Rh))
#
#Residuals:
#    Min      1Q  Median      3Q     Max
#-65.526 -19.139   3.319  21.395  50.545
#
#Coefficients:
#                Estimate Std. Error t value Pr(>|t|)
#(Intercept)     127.1220     5.2640   24.15   <2e-16 ***
#b1(R, Rref, Rh)  -2.4035     0.1212  -19.84   <2e-16 ***
```

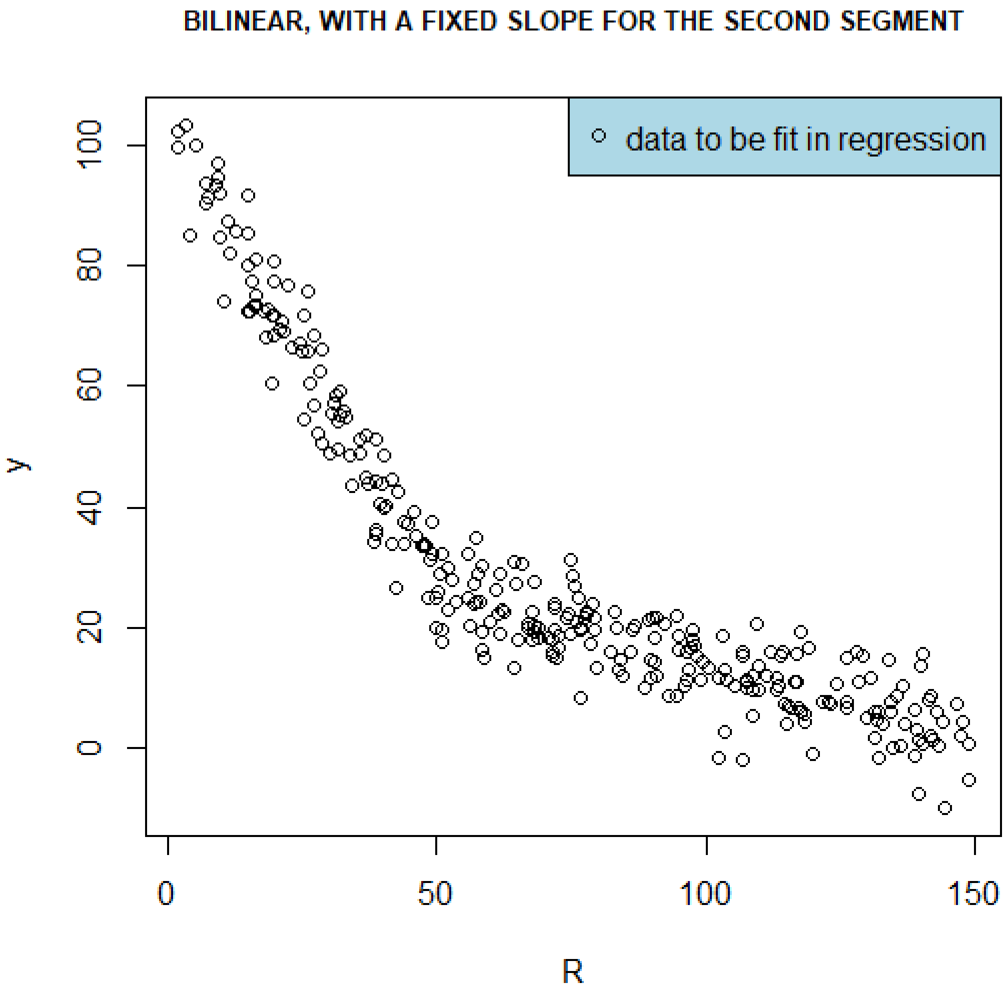
```
#---
#Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
#
#Residual standard error: 27.54 on 298 degrees of freedom
#Multiple R-squared:  0.569,     Adjusted R-squared:  0.5676
#F-statistic: 393.5 on 1 and 298 DF,  p-value: < 2.2e-16

coef <- coefficients(Model)
#> coef
#    (Intercept) b1(R, Rref, Rh)
#     127.122034       -2.403453



plot(R, yn,  xlab="R", ylab="y", col="black", main=Title)
# Plot actual function
abline(v = Rh)

# Add predictions

#Add constrained basis back, and redefine coefficients by factoring out
common distance term for first line:
#yp <- coef[1] + (coef[2])*b1(R,Rref,Rh) + s2*(Rsorted-Rref)
#but for first line (R<Rh), b1 <-  R-Rref, so
#yp <- coef[1] + (coef[2])*(R-Rref) + s2*(R-Rref)
#and
#yp <- coef[1] + (coef[2] + s2)*(R-Rref)
#so
YatRrefBilinear <- coef[1]
s1Bilinear <- coef[2] + s2
s2Bilinear <- s2
#and
yp <- ifelse(Rsorted < Rh, YatRrefBilinear + s1Bilinear*(Rsorted-Rref),
YatRrefBilinear + s1Bilinear*(Rh-Rref) + s2Bilinear*(Rsorted-Rh))
lines(Rsorted,yp, lwd=2, col="blue")
legend("topright", legend=c("desired function", "data to be fit in
regression", "fitted function"),
        lty=c(NA,1), lwd=c(NA,2), pch=c(1,NA), col=c("black", "blue"),
bg="lightblue")
#-----------------------------------------------------------------------------
-----------
```



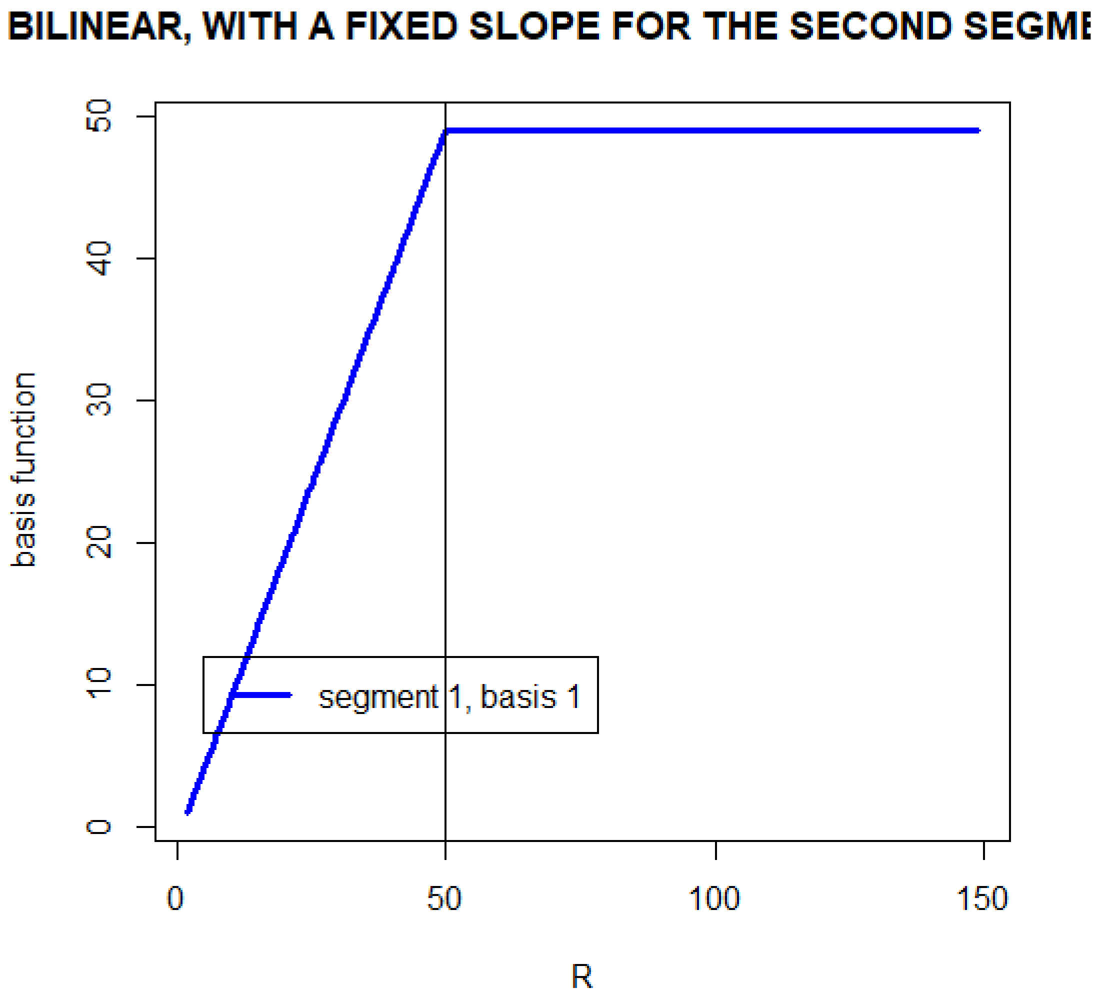
BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGMI

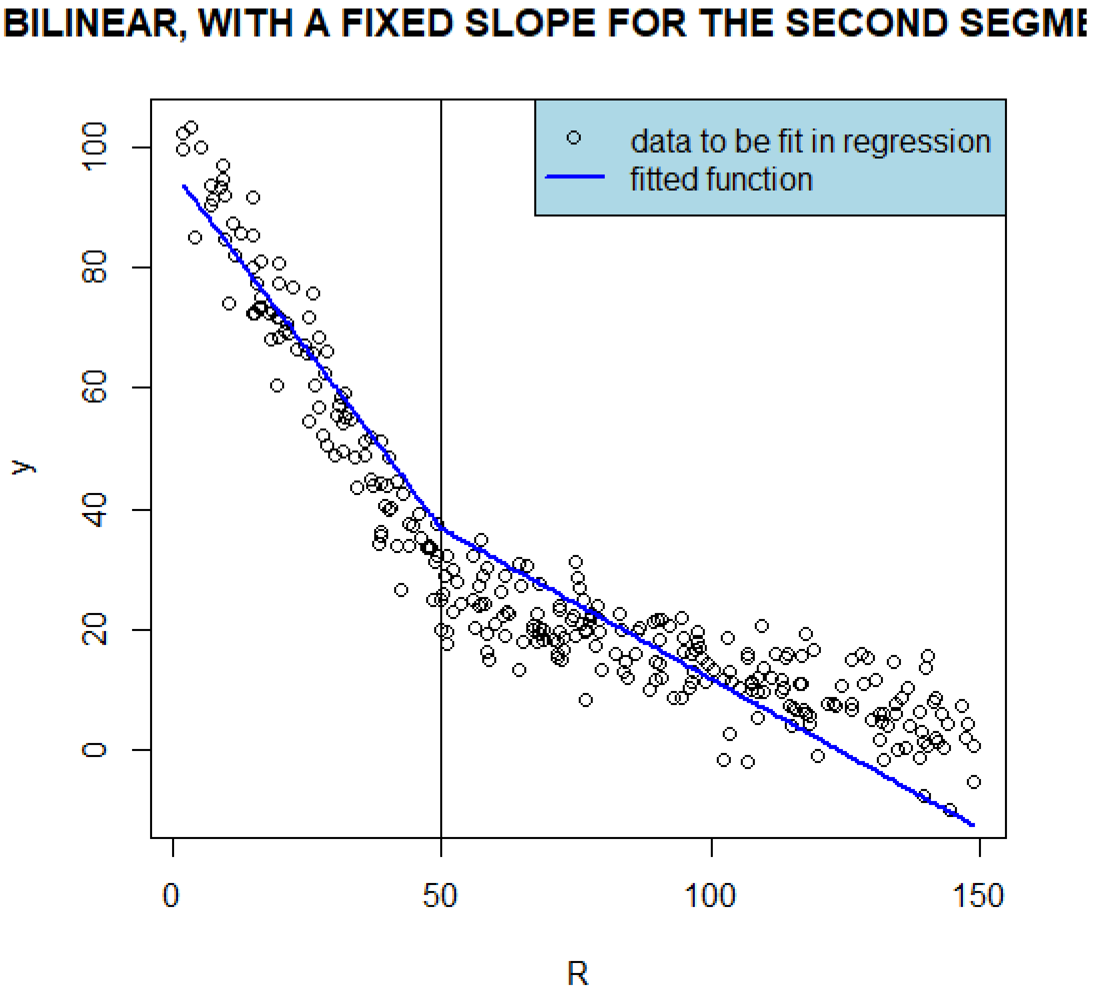
And here is another example, when s2 for the data is -0.25

```
#----------------------------------------------------------------------------
-----------
Title <- "BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGMENT"

# Define function
y <- function(R,YatRref,s1, Rh, s2){
  ifelse(R <= Rh, YatRref + s1*(R-Rref), YatRref + s1*(Rh-Rref) + s2*(R-Rh))
}

# Breakpoints:
YatRref <- 100
Rref <- 1.0
Rh <- 50
s1 <- -1.5
s2Data <- -0.25
s2 <- -0.5

#Generate data (a set without the second segment
set.seed(1)
n <- 300
R <- runif(n, 0, 150)
Rsorted <- sort(R)

# Add some noise:
yn <- y(R,YatRref,s1, Rh, s2Data) + rnorm(n, sd = 5.0)

plot(R, yn,  xlab="R", ylab="y", col="black")
#lines(Rsorted, y(Rsorted, YatRref,s1, Rh, s2),lwd=3, col="red")
title(main=Title, cex.main=0.8)
legend("topright", legend=c("data to be fit in regression"),
        pch=1, col="black", bg="lightblue")
```



```
# Define basis function for each region (a single line)
b1 <- function(x, Rref, Rh){ifelse(x<Rh,x-Rref,Rh-Rref)}

plot(Rsorted, b1(Rsorted,Rref,Rh), xlab="R", ylab="basis function",
  ylim=range(b1(Rsorted,Rref,Rh)), type="l", col = "blue", lwd=3, main=Title)
abline(v = Rh)
legend(5,12, legend=c("segment 1, basis 1"),
        lty=rep(1,1), lwd=rep(3,1), col=c("blue"))
```


BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGME
basis function
0
10
20
30
40
50
0
50
100
150
R
segment 1, basis 1


BILINEAR, WITH A FIXED SLOPE FOR THE SECOND SEGMI
data to be fit in regression
fitted function
y
100
80
60
40
20
0
R
0
50
100
150