Comparison of Two Independent Methods for the Solution of Wave-Scattering Problems: Response of a Sedimentary Basin to Vertically Incident *SH* Waves

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The transient solution to several problems that was obtained by numerical integration of equations of motion using a finite difference (FD) technique is compared with the complex-frequency solutions obtained by the approximate wave theoretical (AL) method of K. Aki and K. L. Larner. The excellent agreement between the two solutions not only provides a comparative check on the accuracies of the two techniques, but also demonstrates that the interpretation of the AL solution is comparable to the Fourier transform of the transient solution premultiplied by an exponential window. Most of the paper is devoted to a discussion of two models that are relevant to the engineering-seismological study of earthquake motions in soft layers of varying thicknesses. The FD and AL solutions show that lateral reverberations of waves produced by the nonplanar structure form complex interference patterns that are not predicted by the usual flat-layer approximations. In one example, constructive interference enhances the peak amplitude of the transient motion over the center of the basin by a factor of 3 relative to the flat-layer solutions. The results indicate that a realistic appraisal of earthquake hazards in areas underlain by soft surficial layers should include the effect of nonuniformity in the structure.

Two methods have been discussed recently for the treatment of seismic-wave propagation in complex structures. One solution uses numerical finite-difference integrations of the equations of dynamic elasticity [*Alterman and Karal*, 1968]; this method (called the FD method here) deals with the propagation of transients in the time domain and is useful for surface-wave problems [*Boore*, 1970]. The other method, called here the Aki-Larner (AL) technique, is a single-frequency solution based on an assumed form for the displacement field [*Aki and Larner*, 1970]. The AL technique is best suited to problems in which body waves impinge on crustal irregularities.

Both techniques are based on approximations and thus it is important to check the accuracy of the solutions. The standard approach is to vary some of the numerical parameters, such as grid sizes in the FD method or the number of scattered waves in the AL method. This approach is useful, but it is more satisfying to compare the results of a given method with the results from a completely different method. Such a comparison was the basic motivation for the work discussed here. By applying the two methods to the same physical problem, a comparative check of the solutions can be made.

Two different problems are considered here: in the first, a layer representing a thin crust with a root overlies a half-space with appropriate elastic constants for the mantle; in the second, a low-velocity sediment-filled basin lies on top of a half-space of more rigid material. Both problems are two-dimensional and involve the seismic motions induced by vertically incident *SH* waves. Besides acting as vehicles for the comparison of the two methods, the problems here are of interest in their own right. This is especially true of the soft-basin problem, a discussion of which forms the bulk of this paper. This problem is directly related to the important one of ground-motion enhancement in sedimentary basins.

DESCRIPTION OF METHODS

AL solution. Although the two methods are more fully described in the literature referenced

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above, a brief discussion is given here. In the AL solution a plane single-frequency wave is assumed to be incident from below. This wave causes a displacement field represented by superposition of plane waves of unknown complex amplitudes propagating in many directions. Inhomogeneous plane waves are allowed. The total motion is an integration over all directions, or equivalently, over horizontal wave number k. Under the assumption that the irregularity in structure repeats itself horizontally, the integral is replaced by an infinite sum. Truncation of this sum and application of the interface conditions yield a system of equations that are solved for the complex scattering coefficients. The equations can then be substituted back into the finite-sum representation of the total displacement to yield the motion and stress at any point in the medium. It is important to note that this is a single-frequency solution.

FD solution. The finite difference (FD) method gives a result in the time domain. In this method a difference equation approximation is first made to the equations of motion and boundary conditions [*Boore*, 1970]. The difference equation is then solved in a recursive manner by using initial displacements and velocities to start the solution. The initial conditions are chosen as required by the specific problem. In the examples presented in this paper analytic values were used to simulate the propagation of vertically incident *SH* waves in the lower half-space. (All the examples involve a layer overlying a half-space.) The transient form of this motion was chosen, for convenience, as a Ricker wavelet [*Ricker*, 1945]. We can require the initial displacements to be far enough from the interface so that the analytical initial displacements are almost zero in the region of heterogeneity (Figure 1). This stipulation justifies our use of simple analytical values to start the problem.

Also shown in Figure 1 are some of the boundary conditions used. Using a symmetric structure and vertically incident *SH* waves enables us to solve for the motion in just half the region, as shown, with the left boundary (plane of symmetry) treated as a free surface. In the FD solution the other vertical boundary and the bottom are artificial boundaries imposed by the requirements of computer storage space. On these boundaries the motion at each time step is given as if the basin were not present. This is

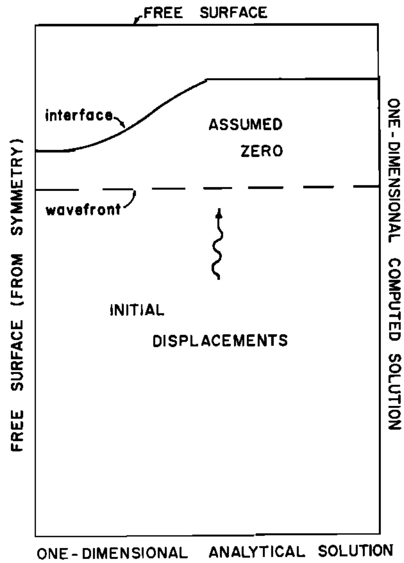

Fig. 1. A schematic diagram giving the general shape of the models considered in this paper, the boundary conditions used at the artificial boundaries, and the location of the initial displacements. The models are symmetric about the left edge.

an approximation and can lead to spurious motion at the surface after the wave, reflected downward from the bump, is again reflected by the impedance mismatch at the artificial boundaries. In practice the artificial boundaries are placed as far from the region of heterogeneity as is economically feasible.

Comparison. To compare the FD solution with the AL solution, Fourier transforms of the time-domain solutions must be computed. There are, however, several practical difficulties in the computation. For one, the spurious disturbances from the grid boundaries can contribute significantly to the later parts of the record. Furthermore, the surface response in some problems reverberates for quite a while, and because the finite-difference solution calculates this response only up to some finite time, noticeable truncation effects may be encountered. Thus it is desirable to apply a window to the time series such that the later amplitudes will be diminished. The question arises as to the type of window that should be applied. According to the convolution theorem, the resulting spectral amplitudes must be compared with a smoothed version of the spectra computed from the AL technique. This, in general, would require a number of AL solutions at nearby frequencies and thus would be cumbersome and expensive. As discussed in an earlier paper [*Aki and Larner*, 1970], however, this smoothing can be conveniently accomplished by computing a single AL solution by using complex frequency. *Phinney* [1965] pointed out that this corresponds to using an exponential window in the time domain with decay time given by the reciprocal of the imaginary part of the radial frequency. Thus by using this particular form for the window, we can directly compare the FD and AL solutions.

CRUST-MANTLE PROBLEM

In this section, we discuss a model in which the slope of the interface is relatively steep; in the section following, a model with a large velocity contrast and a less steep slope is studied. For both models the incident wavelengths are comparable to the dimensions of the heterogeneity, thus precluding an accurate solution by techniques such as simple ray tracing.

Consider the model of a thin (5 km) crustal layer with a root (5 km deep by 20 km wide) overlaying a half-space mantle representation. (Relevant parameters are given at the bottom of Figure 4). This model is not of much intrinsic interest, other than perhaps as a crude representation of a volcanic island root. It was chosen for computational purposes since the FD method is most suitable for calculations of fields within several wavelengths of scattering heterogeneities. The AL method is preferable to the FD method for the more interesting computation of surface motions resulting from the propagation of short-period body waves through continental crusts that have an irregular M discontinuity.

The general nature of the transient solution can be seen in Figure 2. In this figure printer-plot contours of displacement computed by the FD method are shown in vertical sections at different times. The top edge of each plot corresponds to the free surface; the discontinuity is represented by the solid line. These plots are presented only to give a qualitative picture of the displacement variation in space and time. The contours clearly show how the incident transient plane wave is refracted, as expected, into the low-velocity upper medium. Scattered waves in the lower medium are evident. Figure 3 brings out the scattered waves more clearly. Here the contours represent the difference between the solution in Figure 2 and the solution when no bump is present.

A quantitative comparison of the FD and AL solutions is shown in Figure 4 for the period $T = 2.56$ sec. (The incident wavelet had its dominant amplitude near this period.) The results were obtained in the following way: the time series at several sites along the free surface were windowed by multiplication of an exponential with decay time T_D (3.58 sec in this example), and then spectral anomalies were formed by dividing the time series' spectra by the spectra of the similarly windowed solution to the problem with no bump. Also shown in Figure 4 is the dependence of the FD results on the grid spacing DX $(=DZ)$ and on the distances to the artificial boundaries $NX \cdot DX$ and $NZ \cdot DZ$.

The FD solutions appear to converge toward the AL solution as the boundaries are removed and as the grid spacing decreases. In the AL technique, an a posteriori estimate of the error in the surface displacements can be made from study of the match in displacement and stress at the interface [*Aki and Larner*, 1970]. For the problem discussed here, the estimated error is less than 2.5% and the AL solution probably more nearly approximates the true solution than the FD results do. Still, the agreement between the two solutions is quite good, even with the coarser grid, and corresponds to an uncertainty in the results of less than 5%. The largest error occurs near $x = 14$ km and may be caused by a difference in the way the curved and flat interface conditions are treated numerically in the FD method.

SOFT BASIN PROBLEM

An important problem to engineering seismologists is understanding the ground motion near lateral changes in the layer thickness of basins filled with low-rigidity sediments. Because of its inherent interest, the soft basin problem will be considered in some detail. The two related models of the basin that are shown in Figure 5 will be used in an attempt to isolate some of the physical processes involved in the complex wave interference patterns observed in our solutions for motion at the surface of the basin.

Soft basin model 1. The relevant parameters (shear velocities β_1 and β_2 and rigidities μ_1 and μ_2) for this model are shown in Figure 5. The results of the calculations can be applied to other models

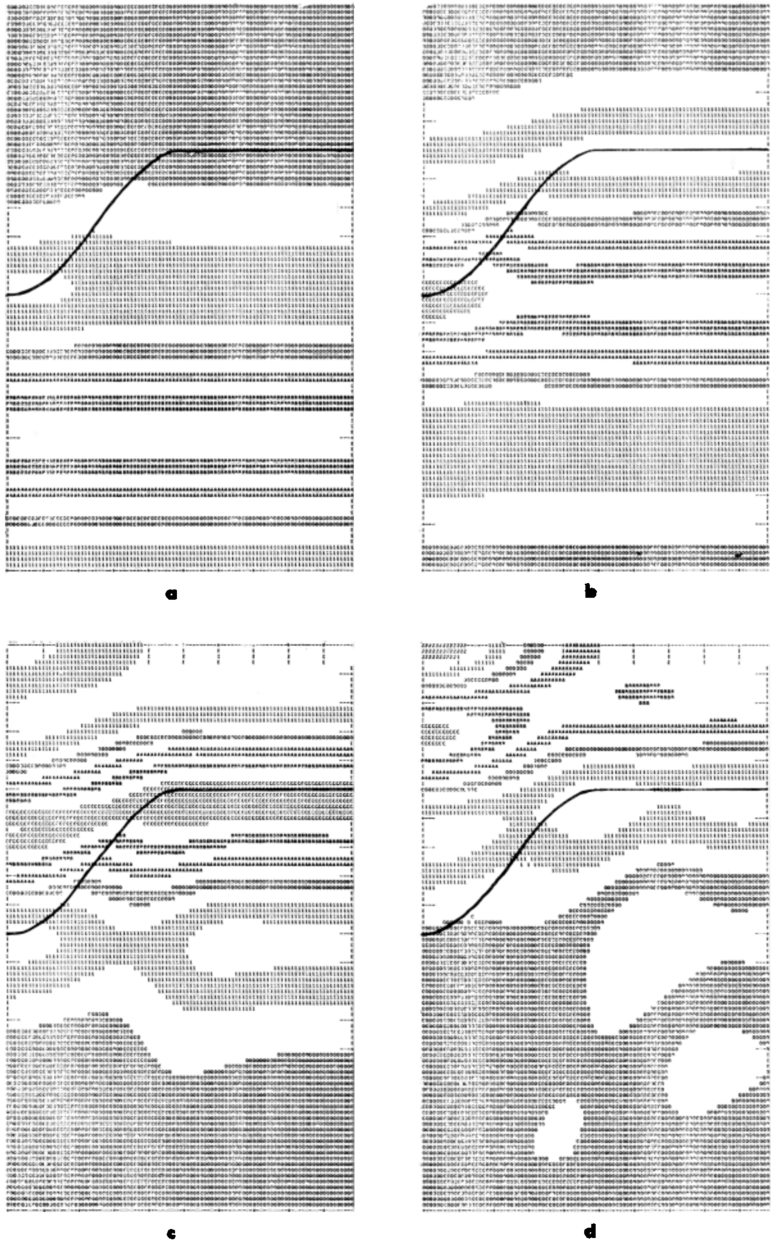

Fig. 2. Contour plots of the total motion in the crust-mantle problem at times (*a*) 1.5, (*b*) 2.5, (*c*) 3.5, and (*d*) 4.5 sec (finite difference solution). See text for explanation.

by scaling distances and times by the same factor. Note that the interface dips less steeply than it did in the crust-mantle model, and the velocity contrast between the layer and the half-space is much larger. Because of this large contrast the layer can, to a first approximation, be thought of as a plate with a rigid bottom and a free upper surface. With this description in mind we chose the layer thicknesses and the dominant wavelengths in the incident pulse such that resonance and anti-resonance conditions will be met at various points along the surface of the

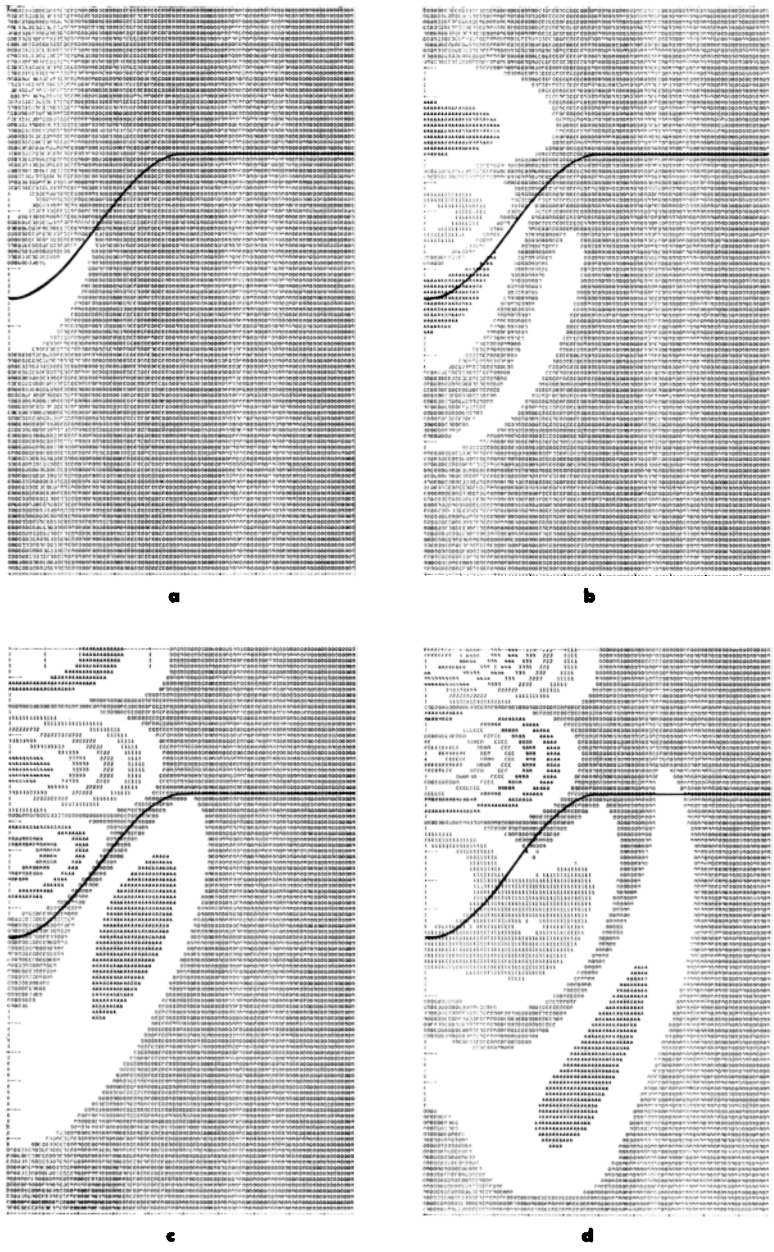

Fig. 3. Contour plots of the scattered motion that were derived by subtracting the solution for the problem of a flat layer with no bump from the solution shown in Figure 2.

basin. Thus a complicated ground motion should be expected. This motion (computed by the FD method) is shown in Figure 6, along with the transient motions expected if the incident wavelet were to propagate through a flat layer of thickness given by the thickness under each recording site on the surface. This reduction of a basically two- or three-dimensional problem to a series of one-dimensional problems is the common approach in engineering seismology studies. It will be termed here the flat-layer approximation (FLA). Although the FLA gives, as predicted,

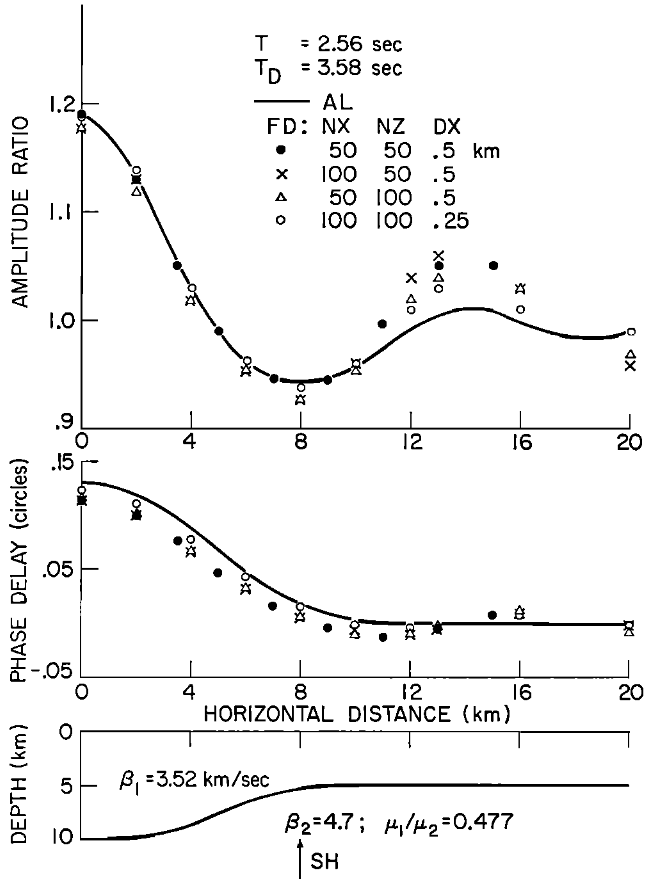

Fig. 4. Comparison of Aki-Larner (AL) and finite difference (FD) solutions at a period of 2.56 sec for the crust-mantle problem depicted at the bottom of the figure. An exponential window of decay time 3.58 sec has been used. The time increment used in the FD solution was 0.025 sec. The shear wave velocities in the layer and the half-space are β_1 and β_2, respectively. The rigidities are μ_1 and μ_2.

a reverberating signal at sites over the basin, a striking discrepancy between the FLA and FD solutions appears at later times. Since the FLA takes vertical interferences into account, the discrepancy is due to the lateral interference caused by the nonplanar shape of the basin. More discussion of these points will be given later. Note, however, the good agreement between the FLA and FD solutions in the first motions ($t < 3$ sec). This shows that the basin's focusing of the first arriving energy is insignificant.

The discussion above assumes that the FD solution is correct. This remains to be shown. As discussed earlier, the first step in assessing the accuracy of the FD solution is to multiply the time traces by an exponential window to eliminate truncation effects and contamination from side boundaries. In the traces shown in Figure 6 the amount of contamination present is slight. This was determined by running the problem twice (once with a grid 7.5 km deep by 6.5 km wide and the other time with a grid 10.0 km deep by 7.5 km wide) and observing the change in the traces. No change was observed in the traces up to a range of 0.8 km. The later part of the trace at $x = 1.2$ km showed a slight change, and the traces for ranges of 1.6 km and beyond showed a significant contamination phase present in the results from the 7.5- by 6.5-km grid. This contamination can be seen after 5.5 sec in the time series at $x = 5.1$ and

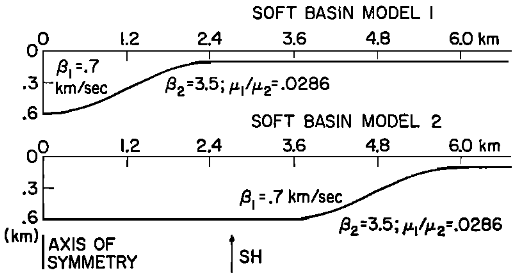

Fig. 5. The two soft basin models discussed in the text.

x = 5.5 km, as shown in Figure 12 for model 2. (This figure was derived from calculations using a grid 7.5 km deep and 10.0 km wide.)

The signals obtained by windowing with an exponential having a decay time of 1.33 sec are shown in Figure 7. The later parts of the record have been effectively diminished, but some of the interesting reverberation has been retained.

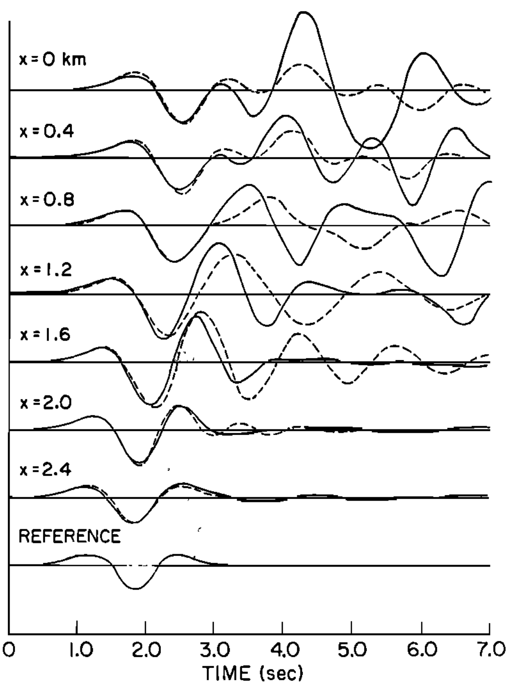

Fig. 6. Computed seismograms for various sites along the free surface over the irregular interface in soft basin model 1. The reference trace is the solution to the auxiliary one-dimensional problem of a constant-thickness layer over a half-space and represents the trace that would be observed far from the basin. The dashed lines are the solutions derived from the flat-layer approximation (FLA). The FD solutions used a grid with Δt = 0.005 sec and $\Delta x = \Delta z$ = 0.10 km.

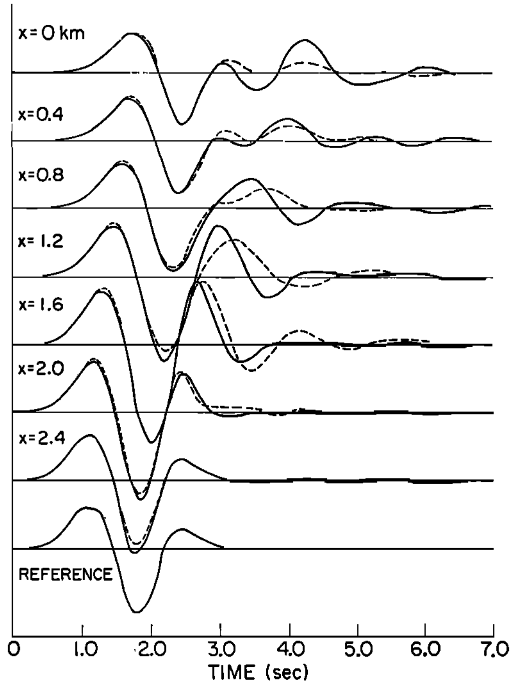

Fig. 7. The results of applying an exponential window having decay time 1.33 sec to the trace in Figure 6.

The next step is to take the Fourier transform of each trace and then normalize the resulting spectrum by the spectrum of the reference trace (the FLA solution at large distance x from the basin). This process is shown in Figure 8 for the trace at x = 0 km. From the plate approximation we expect anti-resonance at frequencies

$$f = (n/2)(\beta/h) \qquad n = 1, 2, \cdots$$

where β is the shear velocity in the layer and h is the layer thickness. The two nulls in the spectrum shown in Figure 8 occur near the predicted values for n = 1, 2.

The amplitude and phase-delay anomalies resulting from the above steps are compared, as a function of station location, with AL solutions at several periods in Figures 9, 10 and 11. The AL solutions were normalized to the Haskell flat-layered solution [*Haskell*, 1960] appropriate to the 0.1-km-thick layer away from the basin. Complex frequency was also used in the Haskell solution. (Because of storage limitations, the FD

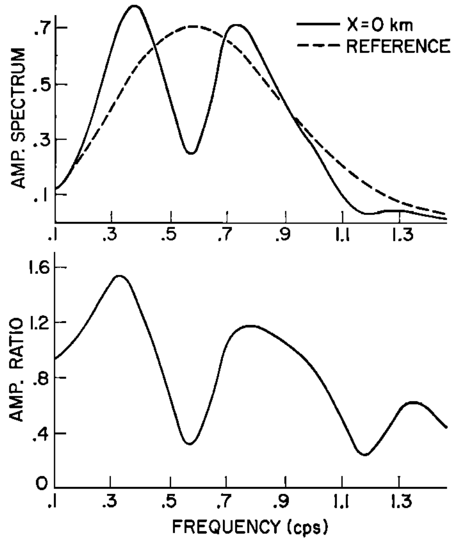

Fig. 8. Amplitude spectra of two time series in Figure 7 and the resulting amplitude ratio. The nulls in the spectrum at $x = 0$ km are primarily caused by anti-resonance in the layer.

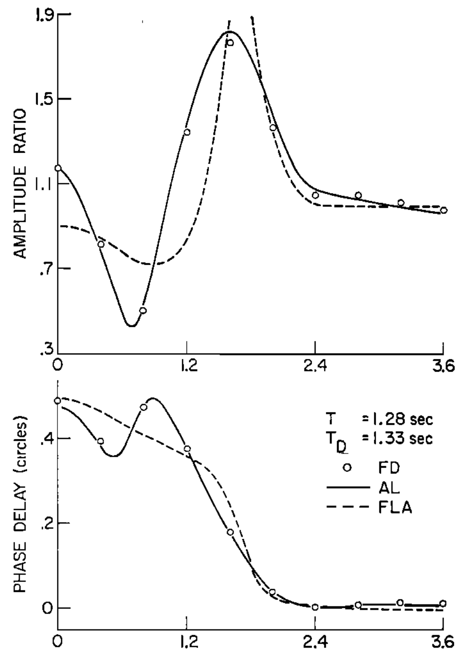

Fig. 9. Frequency domain comparison of the AL, FD, and FLA solutions for soft basin model 1 at the period 1.28 sec.

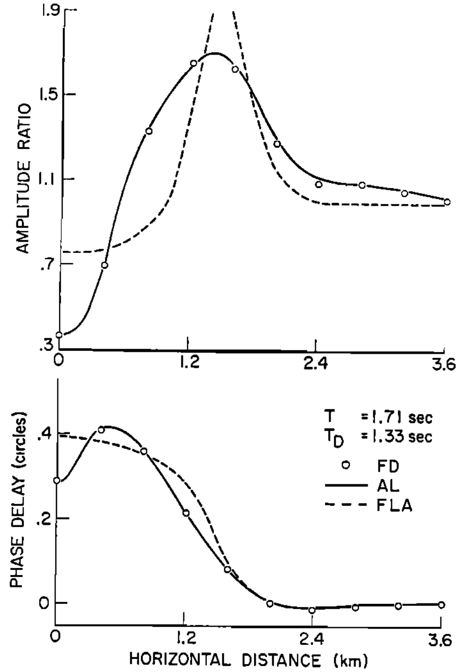

Fig. 10. Frequency-domain comparison of the AL, FD, and FLA solutions for soft basin model 1 at the period 1.71 sec.

results were not accumulated at a large number of spatial locations and thus in the figures could not be connected by a continuous line, as could the AL results.) The agreement between the FD and AL solutions is very good throughout the range of periods illustrated and also holds for other exponential window decay times (not

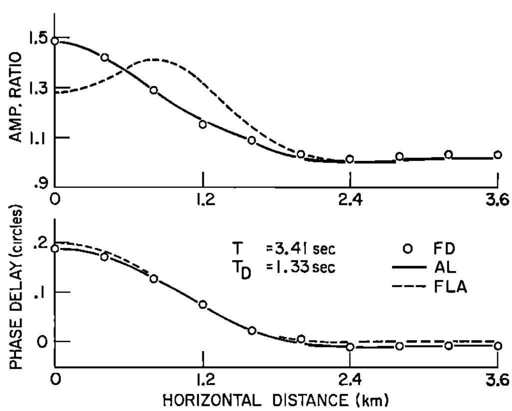

Fig. 11. Frequency-domain comparison of the AL,FD, and FLA solutions for soft basin model 1 at the period 3.41 sec.

shown here); this agreement supports the accuracies of the independent methods and confirms the interpretation of complex frequency as equivalent to windowing in the time domain followed by transformation to the frequency domain.

Soft basin model 2. In an attempt to investigate a monotonic change in layer thickness (as opposed to the confined basin discussed above) the model shown at the bottom of Figure 5 was used. Because of the symmetry about the left-hand boundary, however, the model is one of a basin, but, as will be seen, the scale between the input wave and basin are such that to a first approximation the model is the desired one of a simple change in the thickness of the layer. In fact, the new model is the same as the previous one but with a flat bottom. It was chosen to aid in the physical interpretation, discussed below, of the divergence between the FLA solution and the more exact solutions.

The computed time series at several points along the ground's surface are shown in Figure 12. The input wave was the same as in soft basin model 1. Also, except for the traces at $x = 0$, 2.7, and 3.1 km, the station locations are equivalent, with respect to the layer thickness beneath the station, to those in the previous model.

The FD and AL solutions are compared in Figures 13, 14, and 15 at the same periods and with the same exponential window as in model 1. As before, the comparison is quite good, with agreement to within 5%. This close agreement is expected, of course, because the basic problem has not changed significantly from that in model 1.

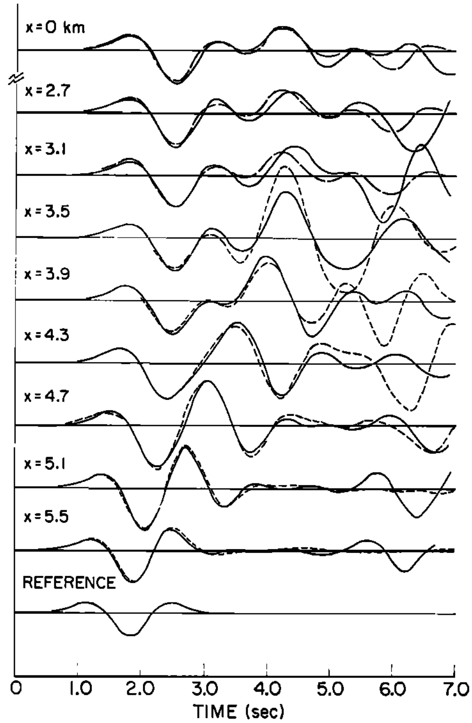

Fig. 12. Computed seismograms for various sites along the free surface over the irregular interface in the soft basin model 2. The long-dashed line for $x = 0$, 2.7, and 3.1 km is the FLA solution. The short-dashed lines for $x \geq 3.5$ km are the FD solutions to the soft basin problem 1. As discussed in the text, the phase in the later part of the $x = 5.1$ and 5.5 km traces is a contamination from the artificial boundaries.

Discussion. With confidence that the time domain and frequency domain solutions are close approximations to the real solutions, we can proceed to discuss the significance and meaning of the results. Probably the most important question concerns how well the results compare with those computed from conventional one-dimensional models based on a flat-layer approximation (FLA) of the two-dimensional structure. The FLA was computed for the models and is compared with the FD and AL results in the appropriate figures. We see that, in the frequency domain, the FLA solution departs from the other solutions in detail, but that in the large sense they agree. The agreement may be artificially caused by the window used, however, for, although the first motions computed from the FLA and FD methods are similar, the later motions are significantly different. Applying a fairly severe exponential window enhances the importance of the earlier arrivals as compared with the later arrivals and thus enhances the comparison between the results. This is clearly shown in Figure 16, where the AL and FLA results for a period of 1.71 sec and a range of window decay times are shown for model 1. As the decay time increases, thus enhancing the importance of the later reverberations, the divergence between the results increases. It is interesting to note that the damping out of later arrivals by use of the window is similar in effect to the propagation of the wave motion through a more realistic anelastic material; thus Figure 16 shows variability analogous to the variability

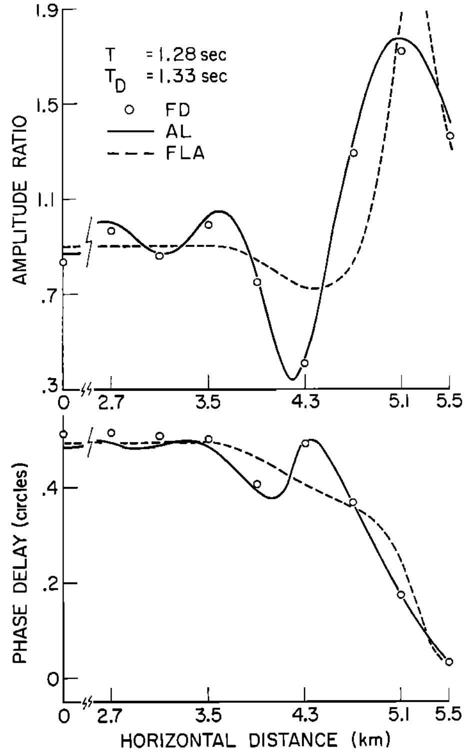

Fig. 13. Frequency-domain comparison of the AL, FD, and FLA solutions to soft basin model 2 at the period 1.28 sec.

that would be produced by a range of Q values.

The departures of the solutions from the FLA are of considerable importance in the prediction of ground motion amplitudes in evaluations of earthquake hazards. It is widely recognized, in practice, that such predictions are complicated by the hysteretic stress-strain response of soils. The results for the soft basin problems indicate that two-dimensional (and presumably three-dimensional) effects add greatly to the complications, especially if the medium allows many reverberations. Even if the medium is strongly attenuating (as represented by the use of T_D = 1.33 sec), the disagreement between observed amplitudes and the amplitudes based on the FLA can be important.

The time domain comparisons with the FLA (Figures 6 and 12) also show significant variations, some of which were not expected from consideration of the spectral amplitude anomalies. Prominent among these is the large pulse in the time series at the center of the soft basin model 1 (Figure 6). This pulse is three times larger than that predicted from the FLA; yet this discrepancy does not show up to such a degree in the amplitude spectra. This variation may be important in earthquake hazard evaluations, for the details of the ground motion's time history may be more important than the relative magnification of its various frequency components in the determination of structural failure; that is to say, the phase spectrum anomaly should also be considered.

The time-domain solutions also aid in the interpretation of the results. Several features must be explained. In Figure 6 three phases seem to be present: the first arrival, the first multiple reflection, and a later arrival (at approximate times 2.4, 4.2, and 5.9 sec in the x = 0 km trace). The first arrival closely follows the FLA solution, but the coincidence for most traces ends there. The first reflection in the x = 0 km trace agrees in position, but not ampli-

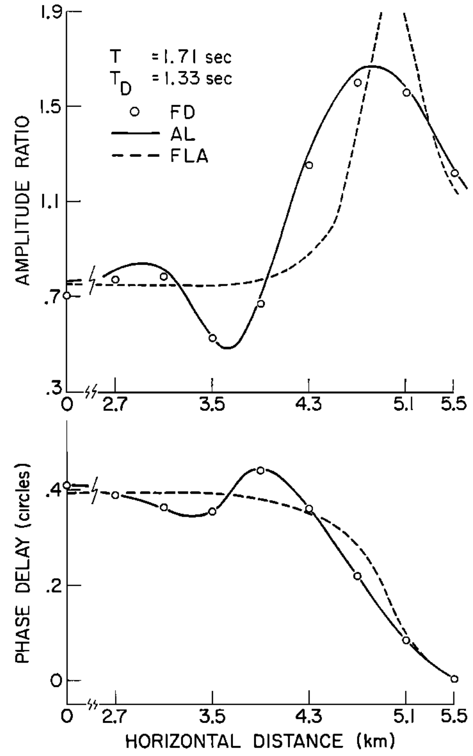

Fig. 14. Frequency-domain comparison of the AL, FD, and FLA solutions to soft basin model 2 at the period 1.71 sec.

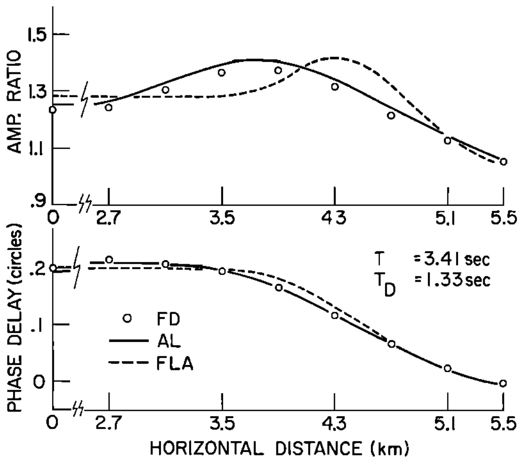

Fig. 15. Frequency domain comparison of the AL,FD, and FLA solutions to soft basin model 2 at the period 3.41 sec.

tude, with the FLA. In other traces between 0.4 and 1.6 km the over-all amplitudes are comparable but the time correlation of the traces is poor. These observations can be explained with reference to the simple ray diagram shown in Figure 17. The difference between the points of arrival on the surface of the direct and first-reflected rays is greatest in the range 0.4 to 1.6 km, and thus the time difference between these arrivals at a given spatial location should be poorly predicted from the FLA. At $x = 0$ km, however, the reflection should arrive as predicted from the FLA since no horizontal refraction takes place. The amplitude of the first reflection in this case is evidently increased by a focusing effect of the first reflections arriving from the sides.

As opposed to the first reflection, which appears to be propagating to the left (toward $x = 0$ km), the third phase appears to travel in the opposite direction. This would imply a source for this phase on the left side of the basin. A comparison of the time traces at equivalent station locations for the two soft basin models (Figure 12) shows that removing the left part of the basin, as is done to a first approximation in model 2, eliminates the third phase from the time series at station locations over the region of decreasing layer thickness. On the basis of these observations, we identify this laterally propagating third phase in model 1 as a continuation of the first multiple reflection from one side of the basin to the other side. This conclusion explains the source of the third phase propagating to the left into the region of constant layer thickness in model 2 (stations for $x < 3.5$ km in Figure 12); except for this phase, the time series for the uniform thickness region are similar to those computed from the FLA.

A comparison of the amplitude anomalies for

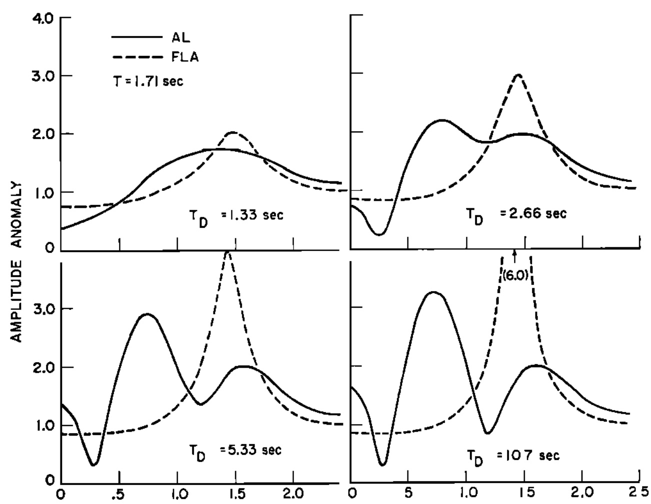

Fig. 16. Comparison of AL and FLA solutions to the soft basin model at a period of 1.71 sec and with various values for imaginary frequency (as expressed by the decay time T_D of the equivalent time-domain exponential window).

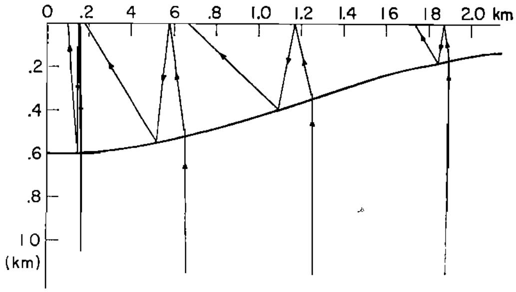

Fig. 17. Geometrical ray paths in soft basin model 1 for four arbitrarily selected rays.

models 1 and 2 shows that, except for the stations over the deepest part of the basin, the results are in substantial agreement. In view of the existence of the third phase in model 1 this agreement may seem puzzling. The exponential window, however, effectively damps the laterally propagating waves from the far side of the basin except near the center of the basin, so that the windowed spectra on one side of the basin do not 'see' the other side.

CONCLUSION

We have shown that, for two different types of problems involving an elastic layer of variable thickness overlying an elastic half-space, the Aki-Larner and finite difference methods give solutions that agree to within better than 5% of one another. Because of the basic differences between the methods, this agreement implies that the computed solutions are close to the real solutions. In making the comparison we have also seen the value of using an exponential window in the time domain and have verified this process as being equivalent to solving the problem in the complex frequency domain. On the basis of the comparison here we can apply, with some confidence, the respective methods to other types of wave scattering problems.

The concurrent use of the frequency- and time-domain solutions was helpful in studying the soft basin problems. The sensitive dependence of the amplitude spectrum on period makes predictions of ground motions difficult; yet the details of the time-domain behavior may be at least as important as the amplification in determining the hazard of seismic energy to man-made structures. We have also seen in these problems that the usual flat-layer approximations give results that can deviate significantly from the actual answers. Thus whenever strong reverberations are expected in a nonplanar layered structure, the flat-layer approximation may be inadequate.

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